Lattice Chern-Simons-Maxwell Theory and its Chirality

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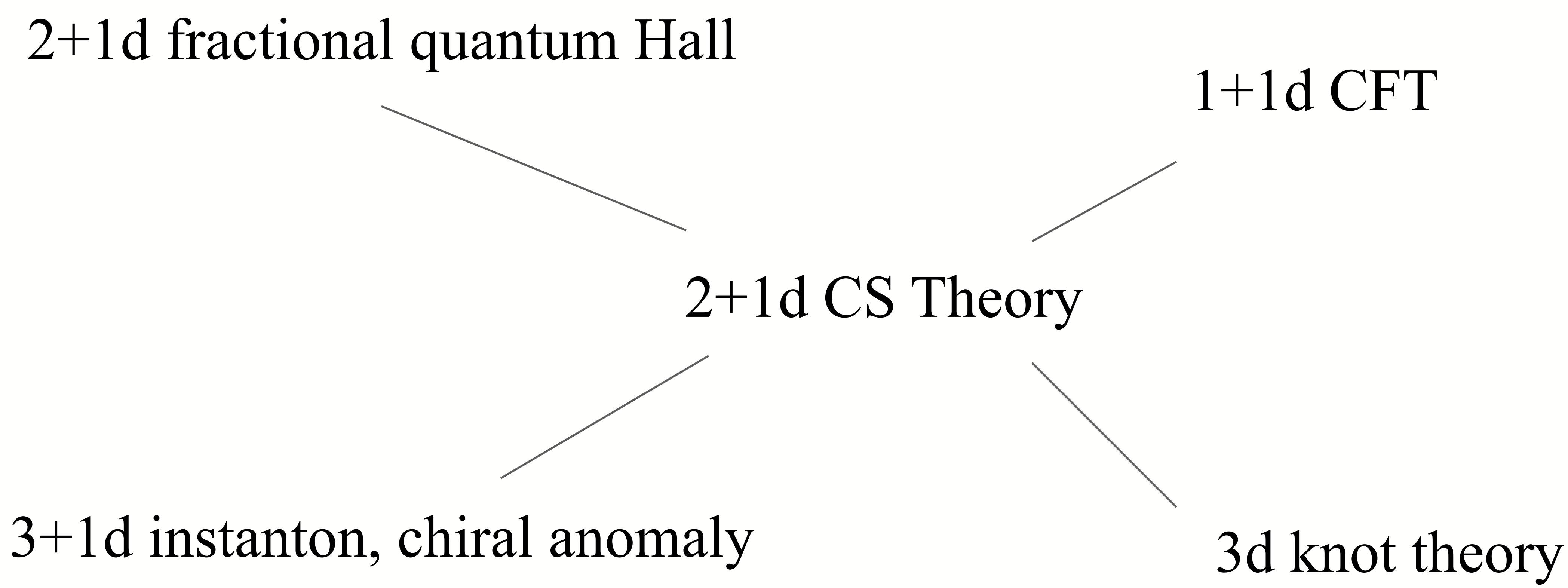

2+1d fractional quantum Hall
1+1d CFT
2+1d CS Theory
3+1d instanton, chiral anomaly
3d knot theory

$$S = \frac{k}{4\pi} \int_{3d} AdA$$

U(1) CS theory, put on lattice

putting on lattice — explicit understanding of the theory

$$AdA = \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu A_\nu$$

$$S = \frac{k}{4\pi}\int_{3d} AdA$$

U(1) CS theory, put on lattice

k even bosonic, *k* odd fermionic

k ground state degeneracy on torus

$1/k$ braiding, $1/2k$ exchange and spin

chiral edge mode

gravitational/framing anomaly

with Ze-An Xu [2410.11034]

$$S = \frac{k}{4\pi} \int_{3d} AdA$$

U(1) CS theory, put on lattice

Two key points

— Not exactly topological, some non-topological term needed (both lattice and continuum)

— Villainized U(1) gauge field for topology

with Ze-An Xu [2410.11034]

$\mathbb{R}$ CS-Maxwell

Villainize to U(1)

Solve the free theory

$\mathbb{R}$ CS-Maxwell

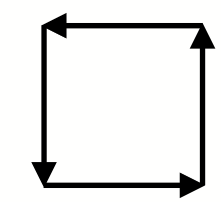
Villainize to U(1)

Solve the free theory

$\mathbb{R}$ CS-Maxwell

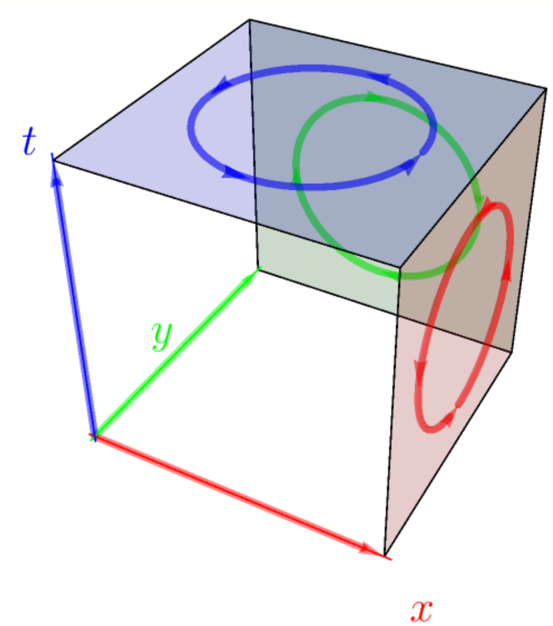
$$i\frac{k}{4\pi}AdA - \frac{1}{2e^2}|dA|^2$$

A real gauge field for now

A_l real field on link l dA_p lattice curl on plaquette p 

Maxwell $$\sum_{\text{plaq. } p} (dA_p)^2$$

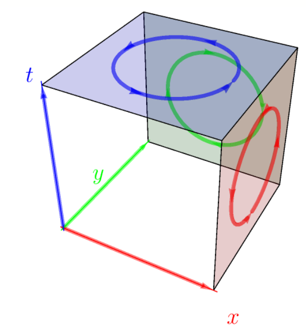
CS $$\sum_{\text{cube } c} (A \cup dA)_c$$



$\mathbb{R}$ CS-Maxwell

Maxwell term necessary,
pure CS term problematic

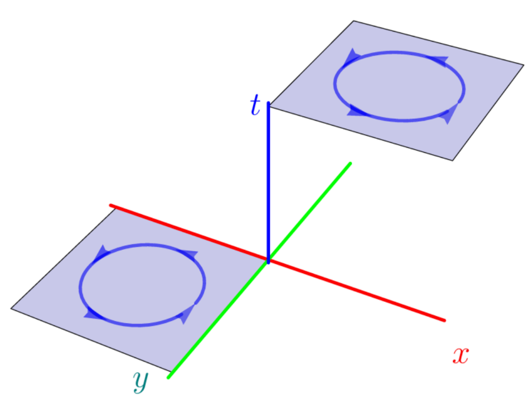
$$i\frac{k}{4\pi}\sum_{\text{cube } c}(A\cup dA)_c$$



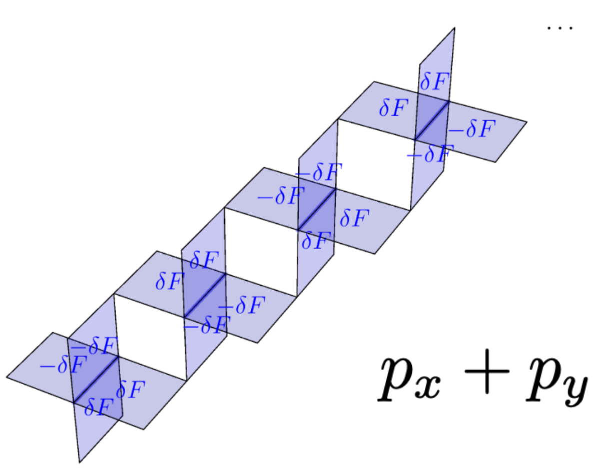
"$\cup\, d.$" has two kinds of zero modes:

— local gauge transformation (expected)

— another non-local unphysical one (problematic!)



$$F_1 + F_2 = 0$$



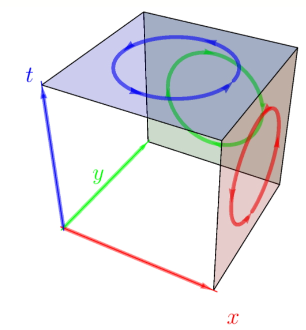
$$p_x + p_y + p_z = \pi$$

Berruto, Diamantini, Sodano, hep-th/0004203

$\mathbb{R}$ CS-Maxwell

Maxwell term necessary,
pure CS term problematic

$$i\frac{k}{4\pi}\sum_{\text{cube } c}(A\cup dA)_c$$



"$\cup d$" has two kinds of zero modes:

— local gauge transformation (expected)

— another non-local unphysical one (problematic!)

— even in continuum, pure CS is ambiguous:
eta-invariant regulator ~ tiny Maxwell term

Bar-Natan, Witten, 1991

$\mathbb{R}$ CS-Maxwell

Villainize to U(1)

Solve the free theory

Villainize to U(1)

A_l real field on link l $\quad\rightarrow\quad$ e^{iA_l} U(1) field on link l

e^{idA_p} U(1) in plaquette p

Villainize to U(1)

A_l real field on link l $\rightarrow$ e^{iA_l} U(1) field on link l

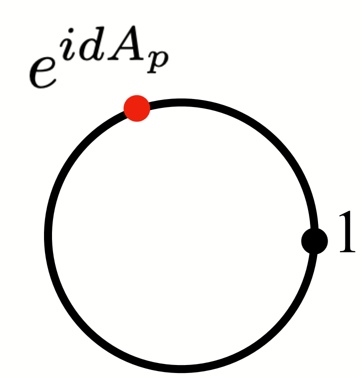
e^{idA_p} U(1) in plaquette p

Villainization:

F_p remains **real** on plaquette p

subjected to $e^{iF_p} = e^{idA_p}$

"how the holonomy around interpolates into the inside"



can parametrize $F_p = dA_p + 2\pi s_p$

$s_p \in \mathbb{Z}$ also dynamical

Villainize to U(1)

parametrize $F_p = dA_p + 2\pi s_p$ $s_p \in \mathbb{Z}$ also dynamical

Villainize to U(1)

parametrize $F_p = dA_p + 2\pi s_p$ $s_p \in \mathbb{Z}$ also dynamical

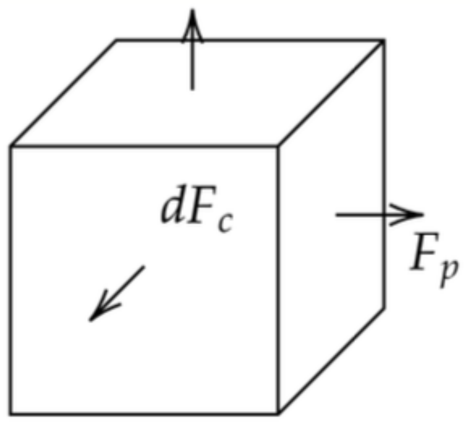
"Dirac string"

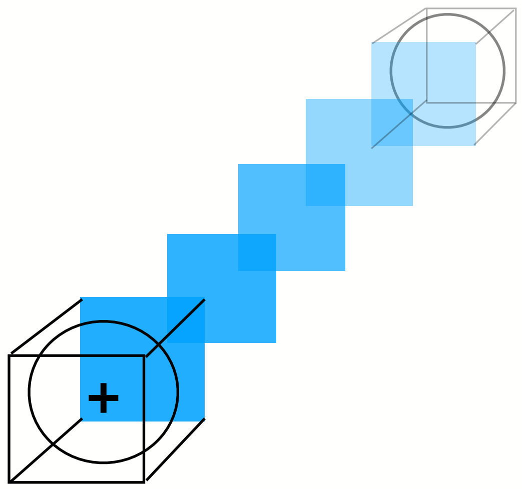
Dirac quantization around 2d surface

$$\sum_{p \in 2d} F_p = \sum_{p \in 2d} 2\pi s_p \in 2\pi\mathbb{Z}$$

monopole defect in 3d

$$dF_c = \sum_{p \in \partial c} F_p = 2\pi ds_c \in 2\pi\mathbb{Z}$$





Villainize to U(1)

parametrize $F_p = dA_p + 2\pi s_p$ $s_p \in \mathbb{Z}$ also dynamical

"Dirac string"

e^{iA_l} is U(1), so A_l is ambiguous by $2\pi\mathbb{Z}$

F_p is well-defined real field since $A_l \to A_l + 2\pi n_l$
$s_p \to s_p - dn_p$

So Maxwell term is well-defined.

How about $(A \cup dA)_c$? $CS = \frac{1}{2\pi}(A \cup dA + A \cup 2\pi s + 2\pi s \cup A)_c$

ambiguous by $2\pi\mathbb{Z}$

Villainize to U(1)

$$CS = \frac{1}{2\pi}\left(A \cup dA + A \cup 2\pi s + 2\pi s \cup A\right)_c$$ has $2\pi\mathbb{Z}$ ambiguity

$$\frac{k}{2} CS$$ has $2\pi\mathbb{Z}$ ambiguity for even k — bosonic CS

Villainize to U(1)

$$CS = \frac{1}{2\pi}\left(A \cup dA + A \cup 2\pi s + 2\pi s \cup A\right)_c$$ has $2\pi\mathbb{Z}$ ambiguity

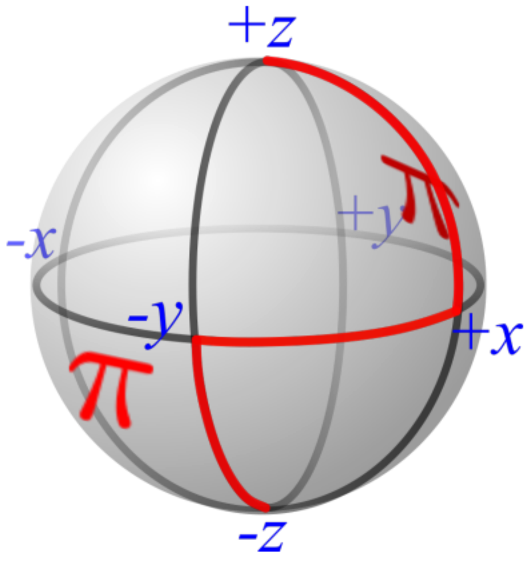
$\frac{k}{2}CS$ has $2\pi\mathbb{Z}$ ambiguity for even k — bosonic CS

has $\pi\mathbb{Z}$ ambiguity for odd k

absorb by a fermion path integral
of Majorana moving along s_p mod 2 — fermionic CS

Gu, Wen; Gaiotto, Kapustin 2014-15

JYChen 1902.06756 for cubic lattice
with Berry phase interpretation



Villainize to U(1)

field config forbid monopole, U(1) EM global symm

$$Z = \left[\prod_{\text{link } l} \int_{-\pi}^{\pi} \frac{dA_l}{2\pi}\right] \left[\prod_{\text{plaq. } p} \sum_{s_p \in \mathbb{Z}}\right] \left[\prod_{\text{cube } c} \int_{-\pi}^{\pi} \frac{d\lambda_c}{2\pi}\, e^{i\lambda_c ds_c}\right]$$

$$\exp\left\{-\frac{1}{2e^2}\sum_p F_p^2 + \frac{ik}{4\pi}\sum_c \left[(A\cup dA)_c + (A\cup 2\pi s)_c + (2\pi s \cup A)_c\right]\right\}$$

Maxwell CS even k

odd k with extra fermion path int $z_{\mathrm{f}}[s \bmod 2]$

Explicitly local!

$\mathbb{R}$ CS-Maxwell

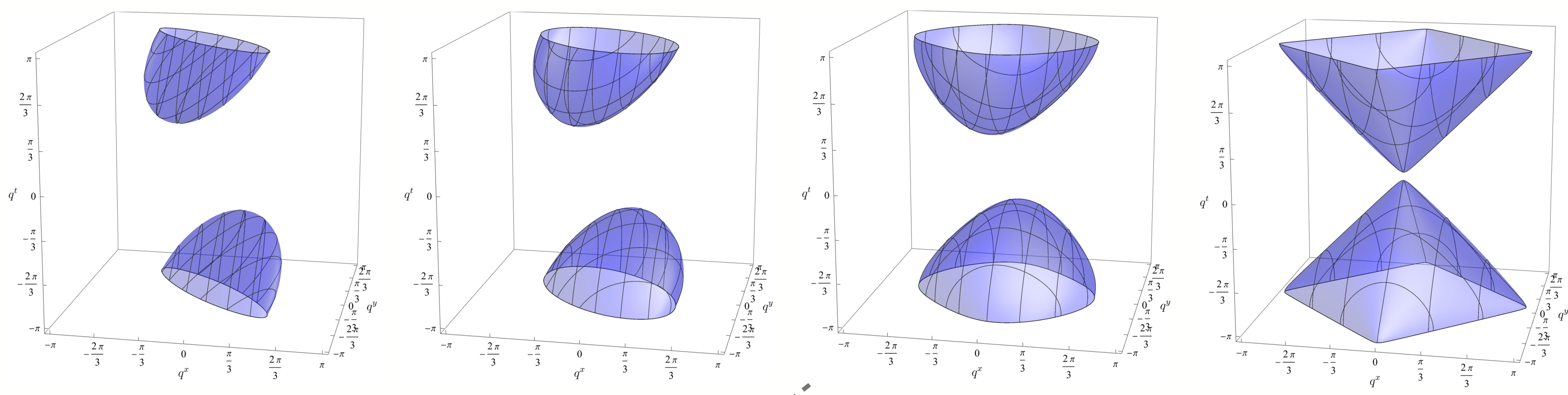
Villainize to U(1)

Solve the free theory

Solve the free theory

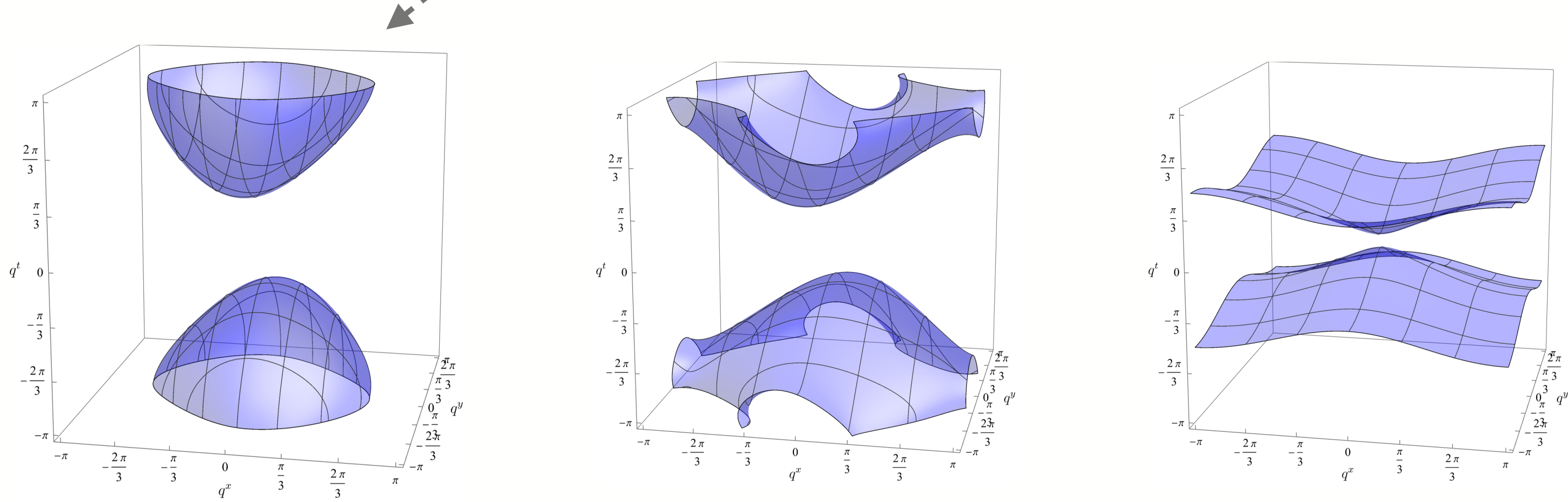
bulk spectrum

(Lorentzian; Fourier transform)



O(1) gap with
tiny Maxwell

increase $1/e^2$



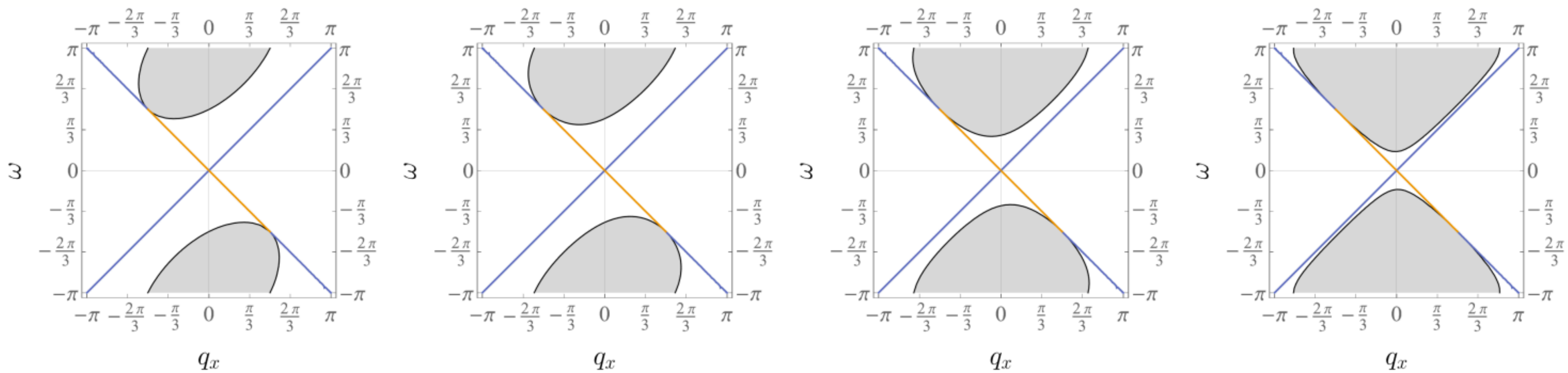
increase *c*

more like band theory

Solve the free theory

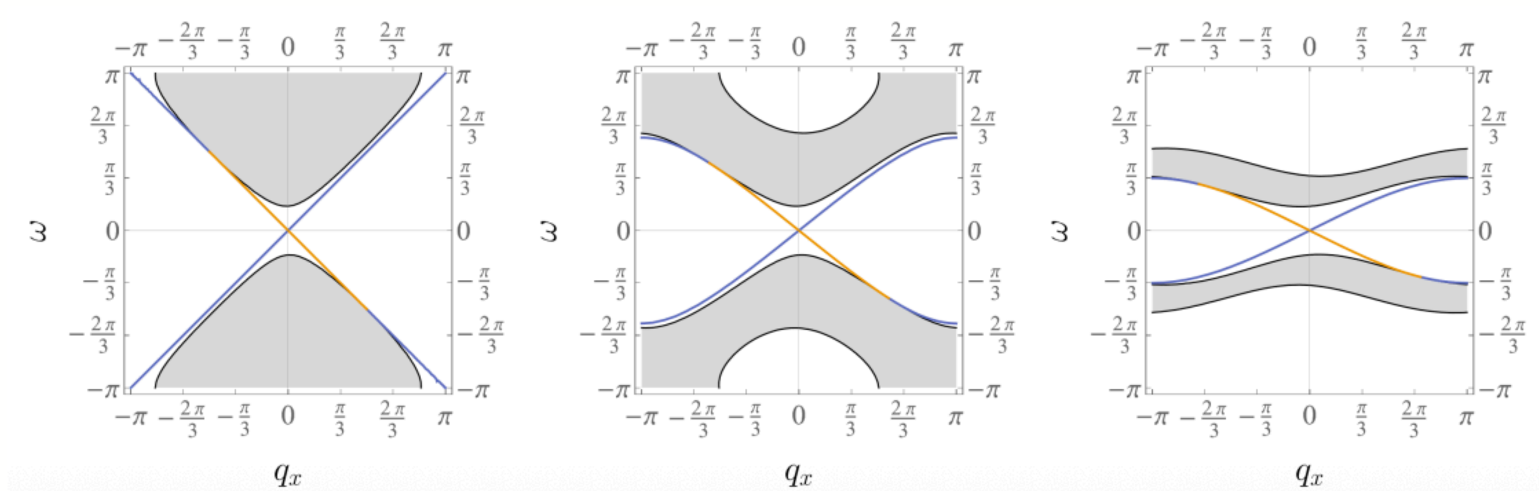
chiral edge spectrum

(Lorentzian; Laplace transform)



O(1) gap with tiny Maxwell

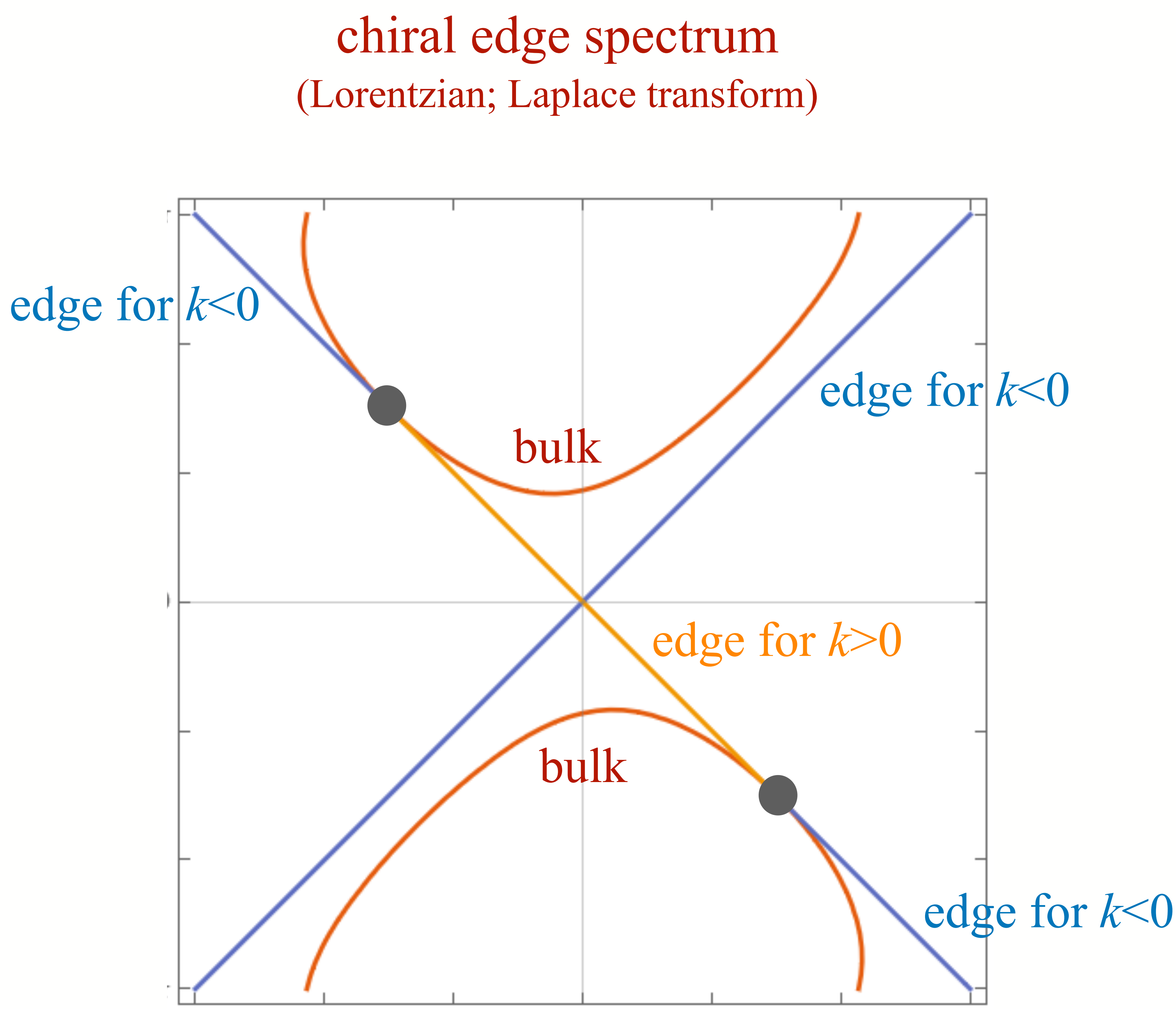
increase $1/e^2$



increase c

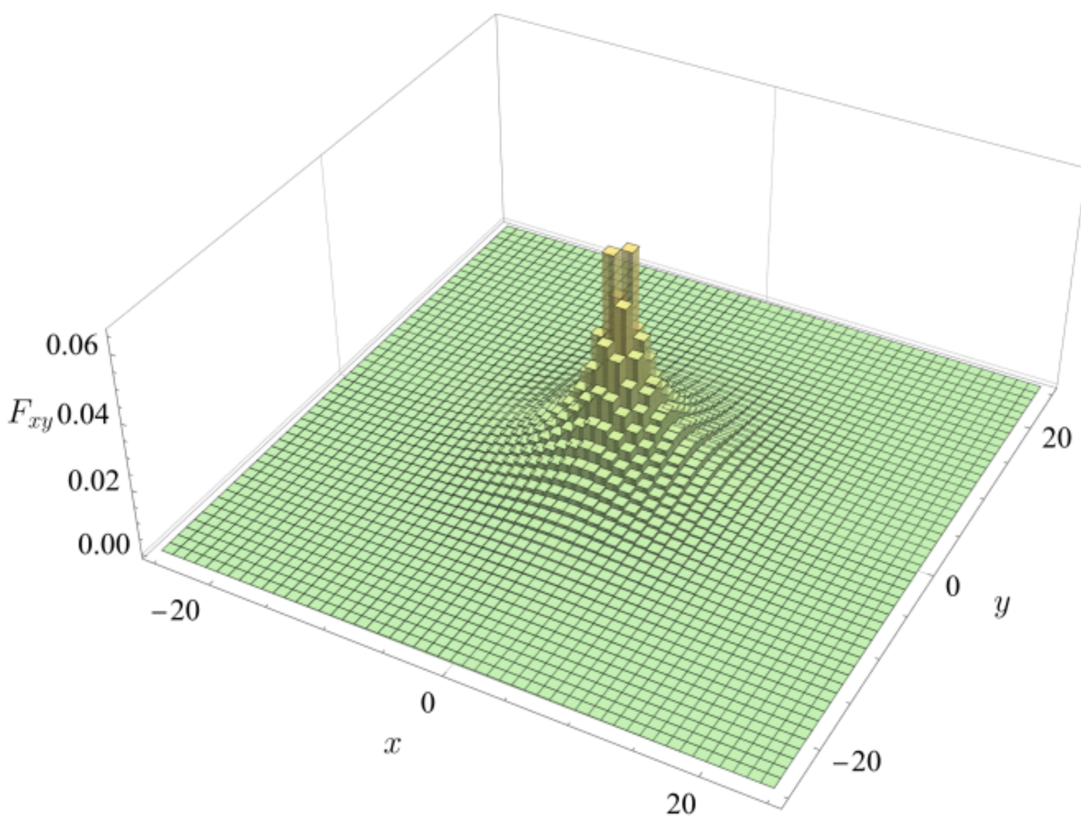
more like band theory

Solve the free theory



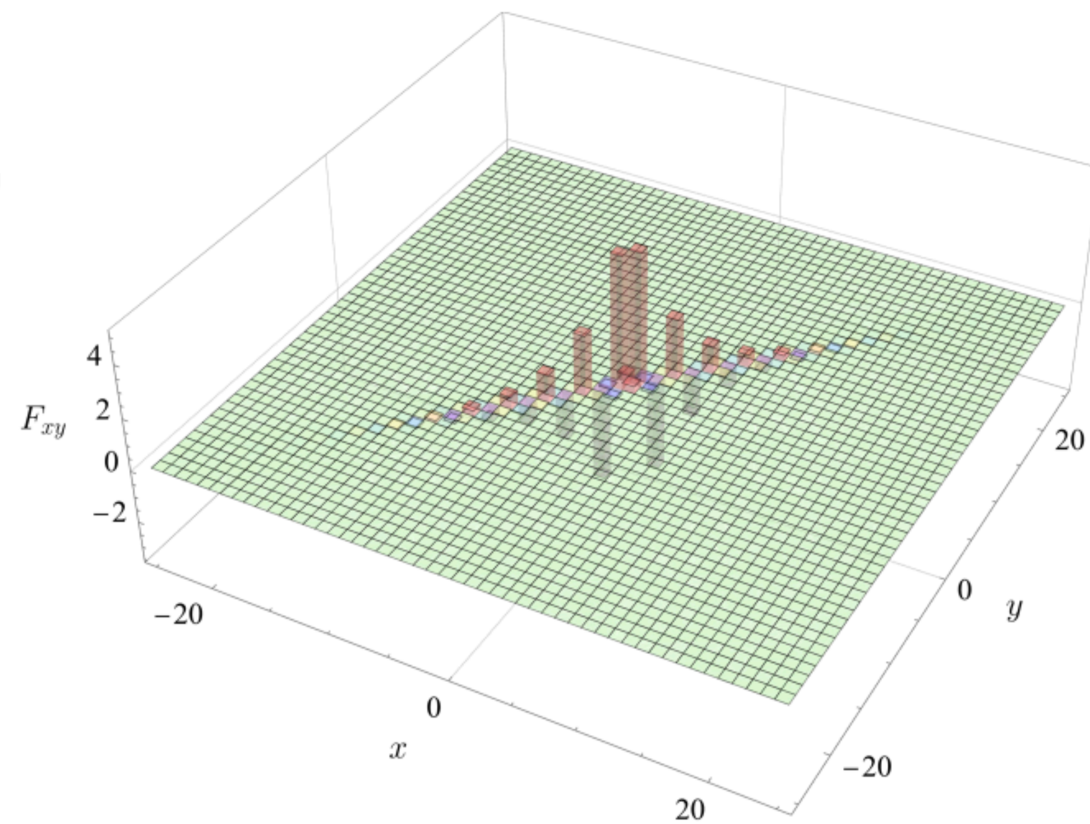
Solve the free theory

Wilson line flux attachment
(Lorentzian)



(a) $e^2 = 1$

geometric framing



(b) $e^2 = 64$

point-split framing

Solve the free theory

ground state degeneracy
(Euclidean)

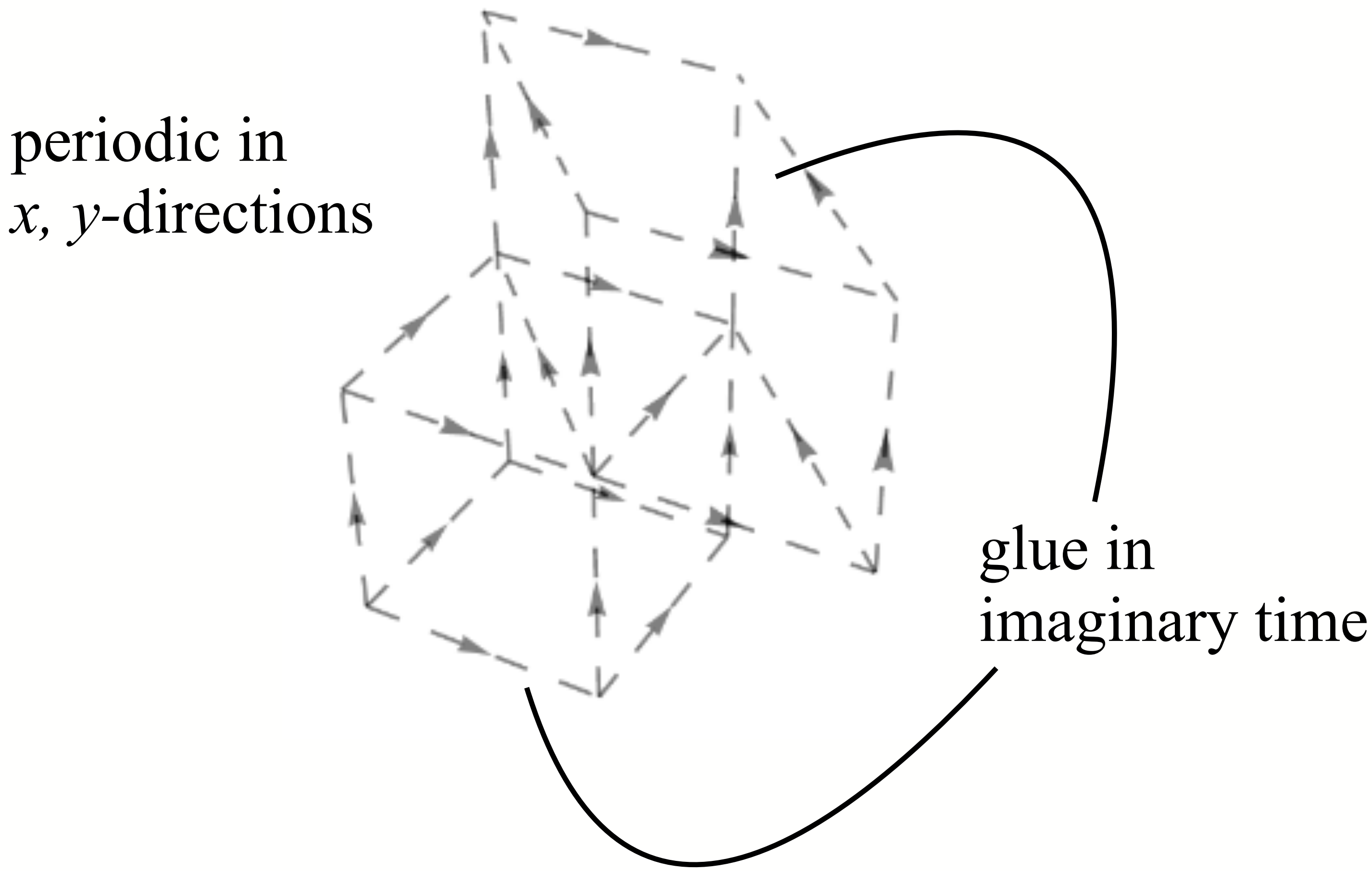
As expected $Z_{(S^1)^3} = |k| \cdot e^{-\text{const}\cdot\text{Vol}}$

because it represents $\mathbf{Tr}(e^{-\beta H})$

const. is ground state energy density

Solve the free theory

gravitational/framing anomaly
(Euclidean)

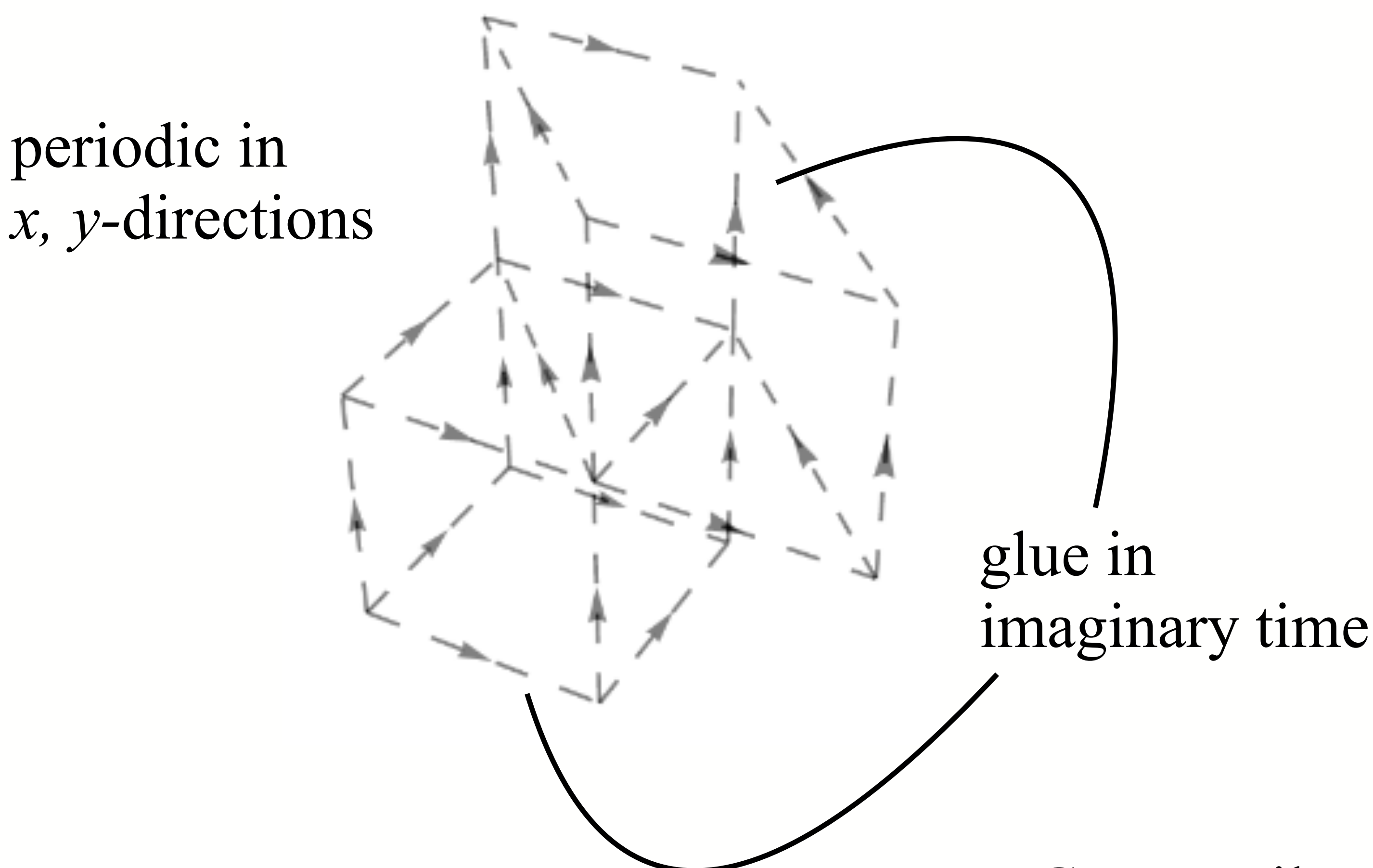


bulk version

$$\mathrm{Tr}(Te^{-\beta H})$$

Solve the free theory

gravitational/framing anomaly
(Euclidean)



bulk version

$$\mathrm{Tr}(Te^{-\beta H})$$

Gauss-Milgram sum of spins of ground states

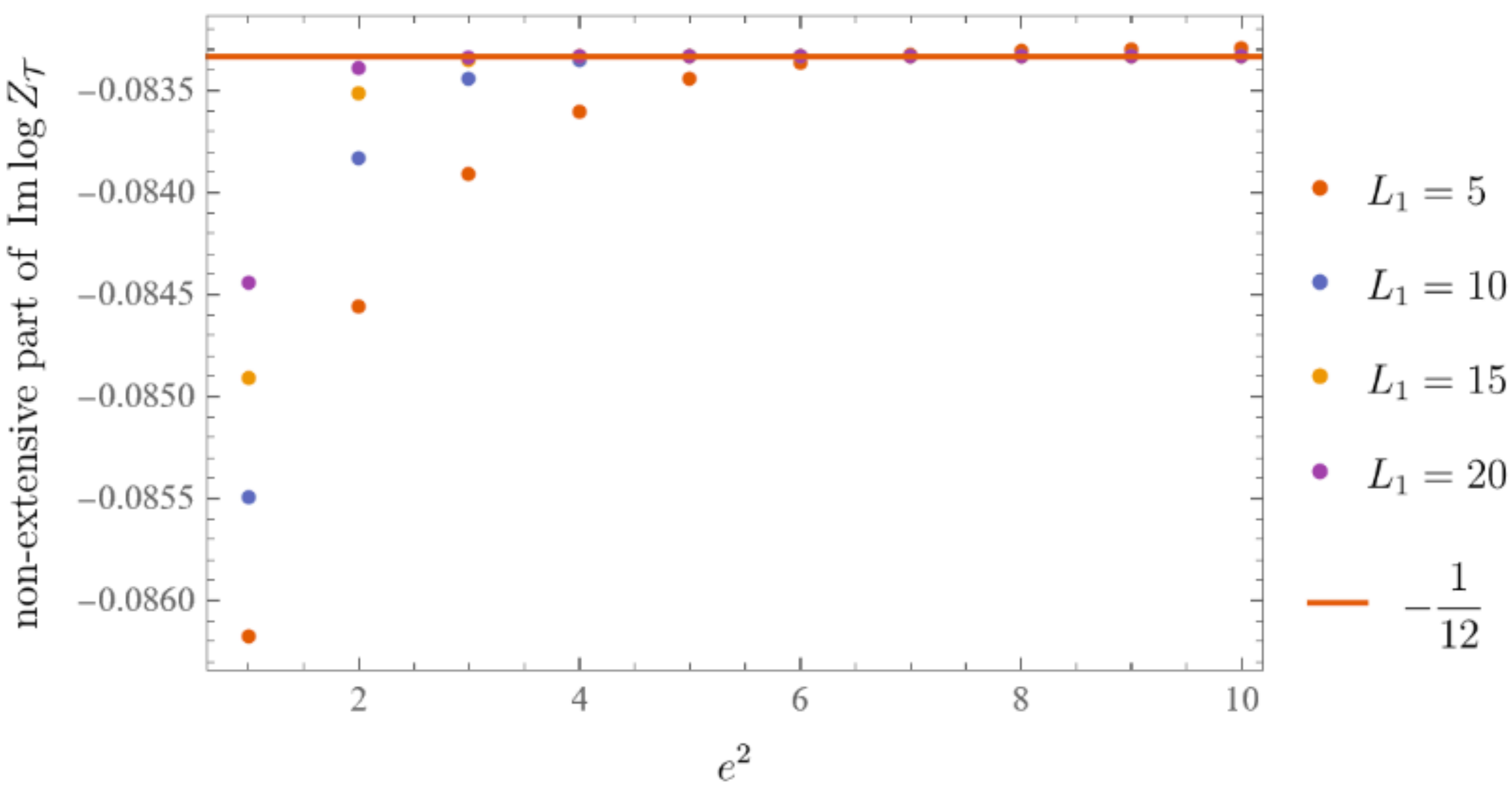
Framing anomaly

$$\langle T \rangle = \frac{e^{i2\pi \mathrm{sgn}(k)(1/8 - 1/24)}}{\sqrt{|k|}}\, e^{-\alpha L^2 + \cdots}$$

with Ze-An Xu, in preparation

Solve the free theory

gravitational/framing anomaly
(Euclidean)



Gauss-Milgram sum of spins of ground states

Framing anomaly

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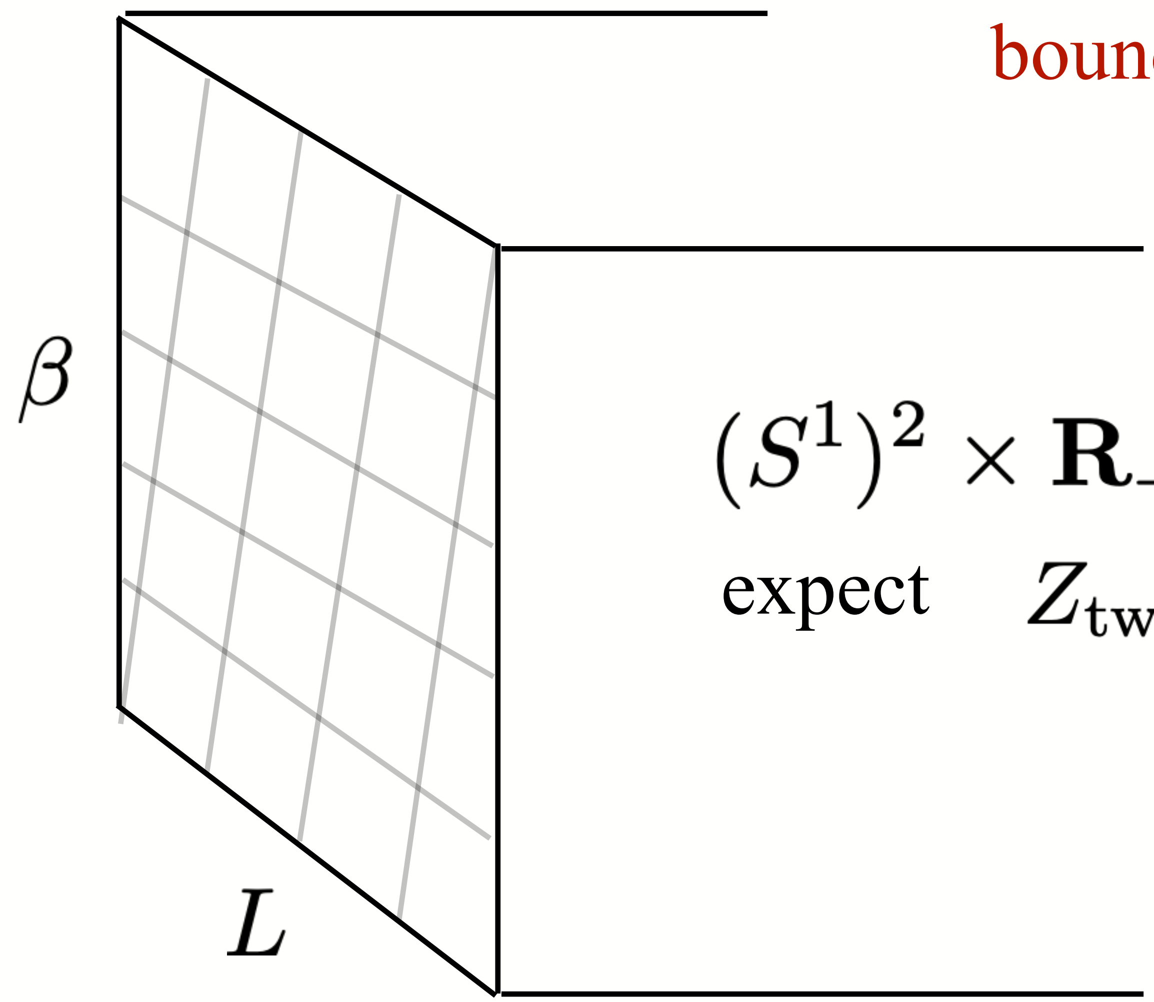
with Ze-An Xu, in preparation

Solve the free theory

gravitational/framing anomaly
(Euclidean)

boundary version



$$(S^1)^2 \times \mathbf{R}_+$$

expect $Z_{\text{twist}} = e^{i \frac{2\pi}{24} \frac{1}{\beta}} Z$

~ pick up the "negative energy chiral edge modes"

"linear dispersion" gives $1 + 2 + 3 + \cdots = -1/12$

$$\sum_n \frac{1}{N} \frac{n}{N} f\left(\frac{n}{N}\right) = \int_0^\infty dx\, x f(x) - \frac{1}{12N^2} + \cdots$$

Solve the free theory

k odd fermionic, k even bosonic

k ground state degeneracy on torus

$1/k$ braiding, $1/2k$ exchange and spin

chiral edge mode

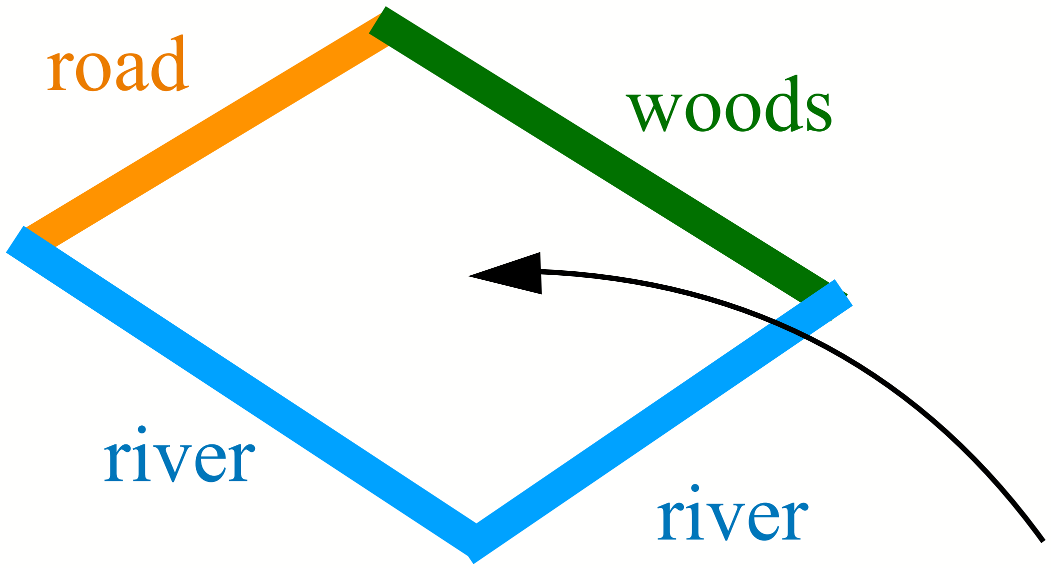
gravitational/framing anomaly

How about non-abelian

How about non-abelian

categorical generalization of Villainization

Wilson, Berezinskii,… in physics ~~~~~~ Kan, Quillen, Grothendieck… in math category theory is much like some kind of board game:



Which types of castle are allowed to play here?

JYChen [2406.06673]
with Peng Zhang [2411.07195]

Category theory in physics, but want:

from IR TQFT/CFT to generic QFT at generic scale
from discrete d.o.f. to continuous d.o.f.

Lattice Chern-Simons

Lattice QCD simulation
of Yang-Mills instanton

A more systematic relation
between lattice & continuum QFT