A study of 1D critical lattice models with exact fusion category symmetry

Chenjie Wang

University of Hong Kong

S.Q. Ning, B.B. Mao, CW, SciPost Phys. 17, 125 (2024)
S. Roychowdhury, CW, work in progress

100 Years of Quantum Mechanics, ICISE, Quy Nhon, 10/2025

Topological phases

(anyons, fractional statistics, …)

generalized symmetry

Quantum critical phenomena

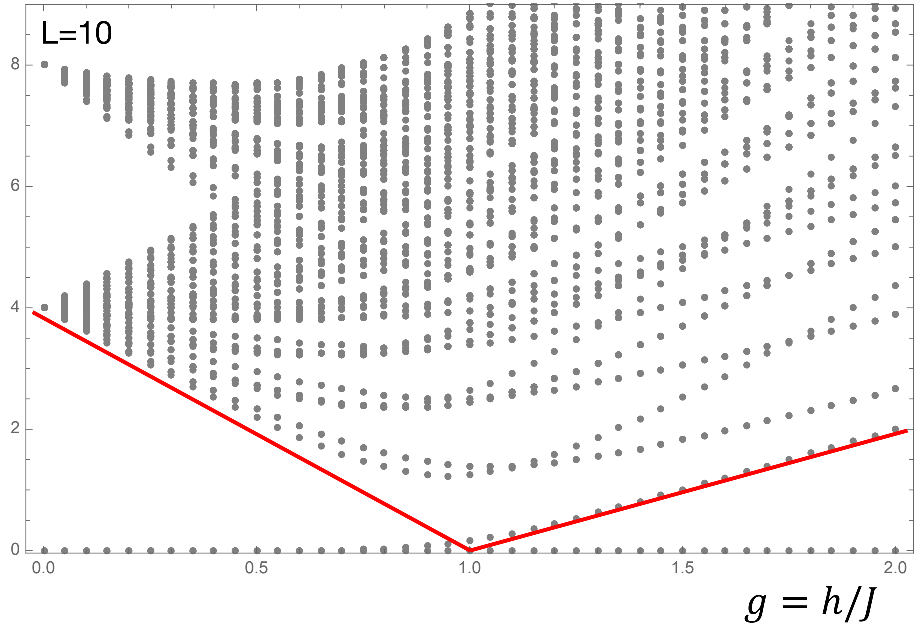
(universality class, critical exponents, …)

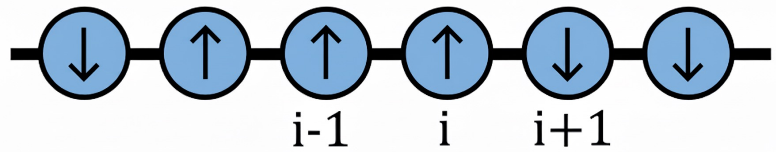
Outline

- Introduction to categorical symmetry

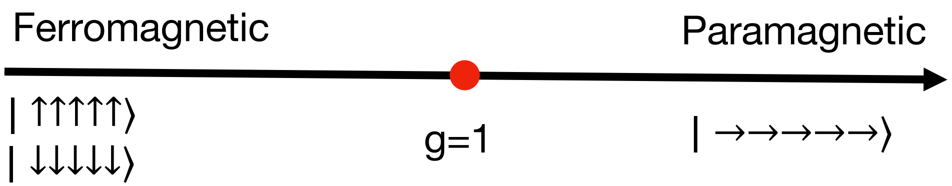
- Building lattice models with exact categorical symmetry and some numerical study

1D transverse-field Ising model





$$H_{\text{Ising}} = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x$$



Spin-flip symmetry:

$$X = \prod \sigma_i^x$$

$$HX = XH$$

$$\sigma_i^z \to -\sigma_i^z, \qquad \sigma_i^x \to \sigma_i^x$$

Kramers-Wannier duality

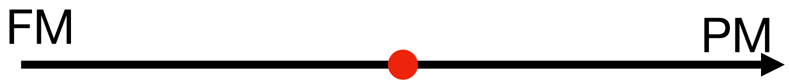
Non-invertible operator: $D \times D = (1+X)T_{\text{trans}} \sim 1 + X$

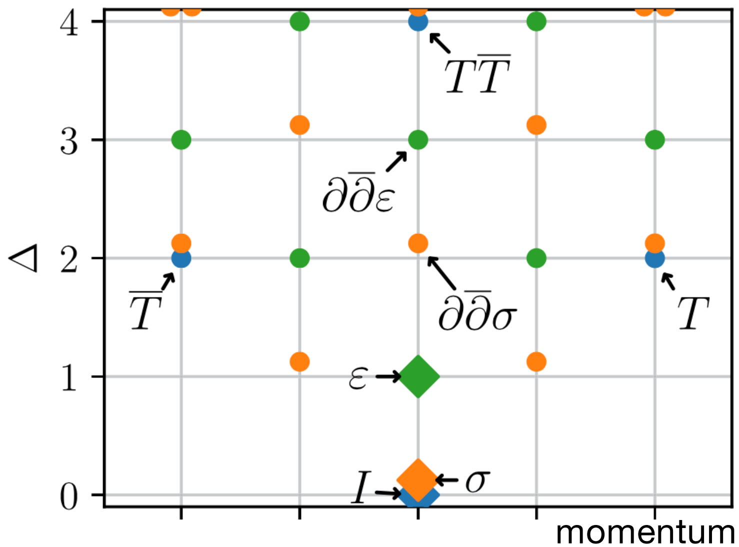
Spin basis		Domain-wall basis
↑↑↑↑↑↑↓↓↓↓↓↓↑↑↑↑↑↑		000001000000100000
$\sigma_i^z \sigma_{i+1}^z$		τ_{i+1}^z
σ_i^x	$\Leftrightarrow$	$\tau_i^x \tau_{i+1}^x$
σ_i^z		Non-local operator
$-J\sum \sigma_i^z \sigma_{i+1}^z - h\sum \sigma_i^x$		$-h\sum \sigma_i^x \sigma_{i+1}^x - J\sum \tau_{i+1}^z$

Criticality is at self-dual point $J = h$

Kramers and Wannier, Phys. Rev. 1941

Symmetry properties of Ising CFT





$$E = \frac{2\pi}{L}\Delta$$

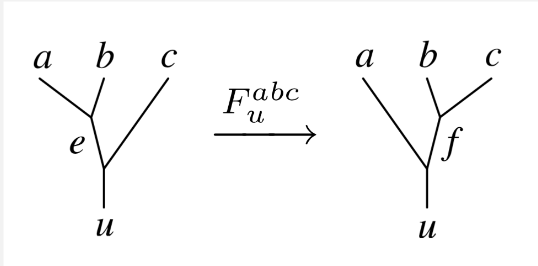
$$c = \frac{1}{2}, \Delta_\sigma = \frac{1}{8}, \Delta_\epsilon = 1$$

$X|\epsilon\rangle = |\epsilon\rangle,\quad X|\sigma\rangle = -|\sigma\rangle$

$D|\epsilon\rangle = -|\epsilon\rangle,\quad D|\sigma\rangle = 0$

$\{1, X, D\}$ or $\{1, \epsilon, \sigma\}$ form a ***fusion category*** (with additional data), instead of a group:

$$X \times D = D \times X = D$$
$$D \times D = 1 + X$$



$$H_{Ising} = H_{critical} + \sum_A \alpha_A^{pert} O_A$$

$O_A \Leftrightarrow |A\rangle$ (State-operator correspondence)

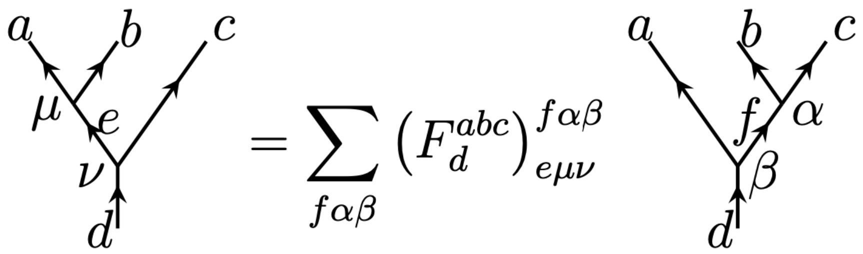
No ***symmetric relevant*** perturbations ($\Delta < 2$) under Ising fusion category symmetry.

Zou, Milsted, Vidal, PRL 2018

Fusion category

✦ A set of operators $\{S_a\}$ follows a fusion algebra $S_a S_b = \sum_c N_{ab}^c S_c$

✦ non-trivial associator



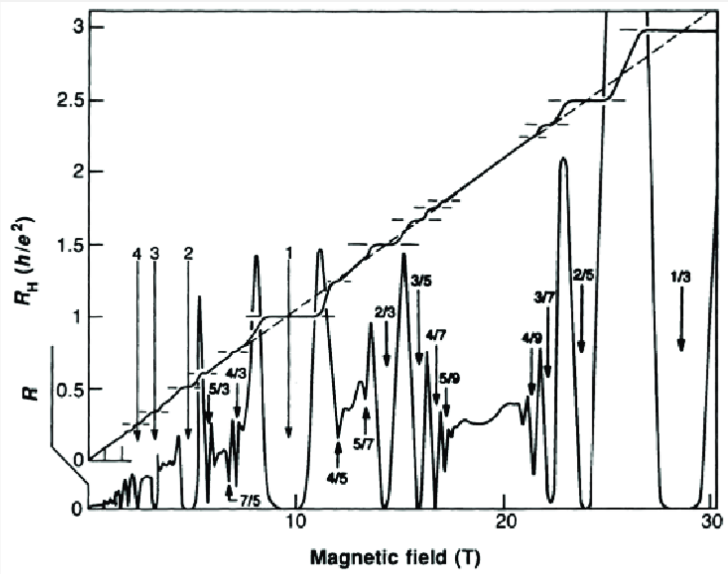
✦ Examples:

- Quantum anomaly of group: $C_G = (G, \nu_3)$, ν_3 is group cocycle (cohomology)
- Ising fusion category $\{1, \psi, \sigma\}$, with $\sigma \times \sigma = 1 + \psi$.
- Fibonacci category $\{1, \tau\}$, with $\tau \times \tau = 1 + \tau$.

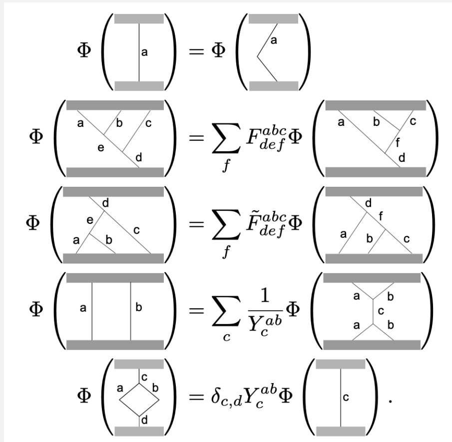
Runkel, Fuchs, Schweigert and coworkers, ~2006, several works; Davydov, Kong, Runkel, arXiv:1107.0495; Bhardwaj & Tachikawa arXiv:1704.02330; Ji & Wen, 2021; Kong-Lan-Wen-Zhang-Zheng, PRR 2020; Apruzzi et al, arXiv:2112.02092; Freed-Moore-Teleman, 2022; many others

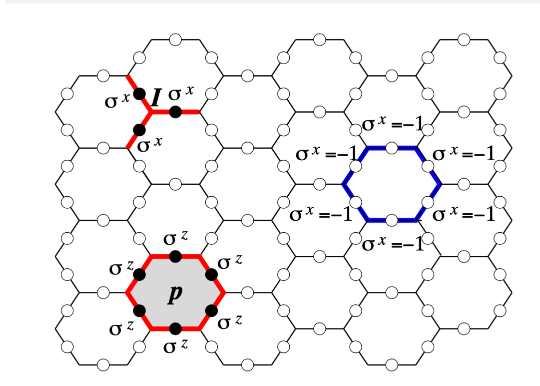
Anyons and topological defects

Fractional Quantum Hall states



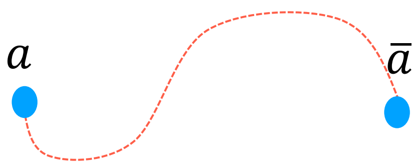
Kitaev quantum double & String-net model

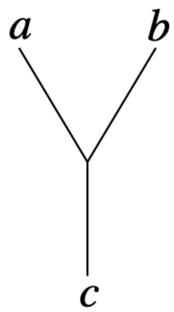


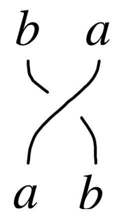


$$H = -\sum A_v - \sum B_p$$

Created by a string operator:

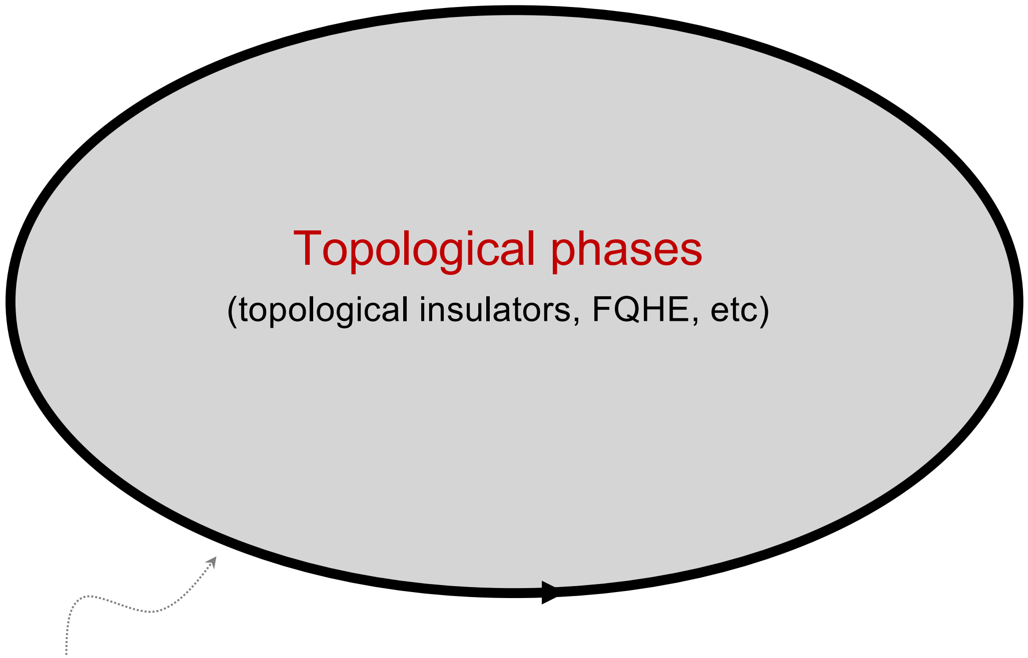






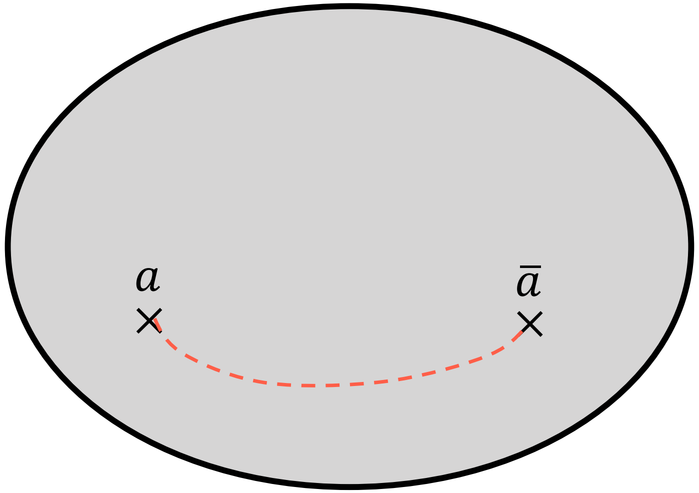
Tsui, Stormer, Gossard'82; Laughlin'83, Kitaev'03; Levin & Wen'05

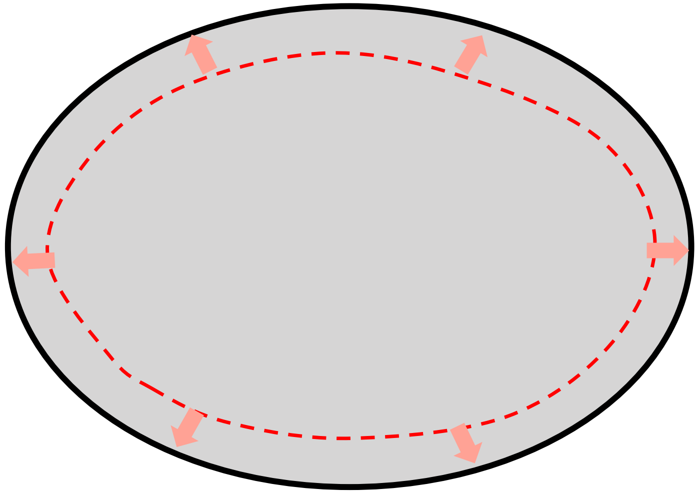
Bulk-boundary correspondence
(topological holography)



Low-energy dynamics on edge $(energy \ll bulk\ gap)$

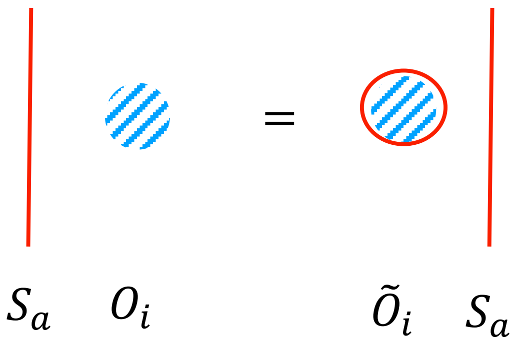
String operators act as symmetry





String operator S_a

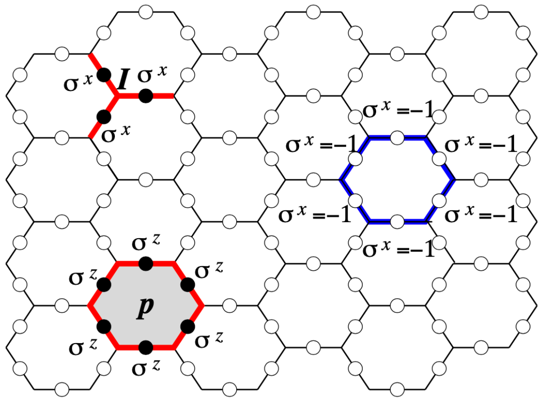
important requirement of many-body system: **locality-preserving**



$$S_a O_i S_a^{-1} = \tilde{O}_i$$

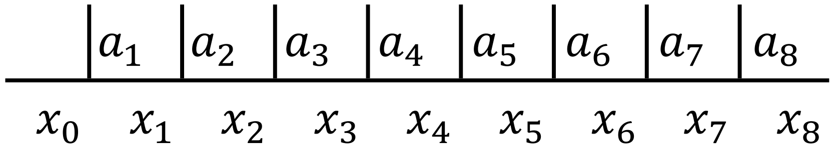
Edge of string-net model

Topological order
(with anyons)



$$H = -\sum A_v - \sum B_p$$

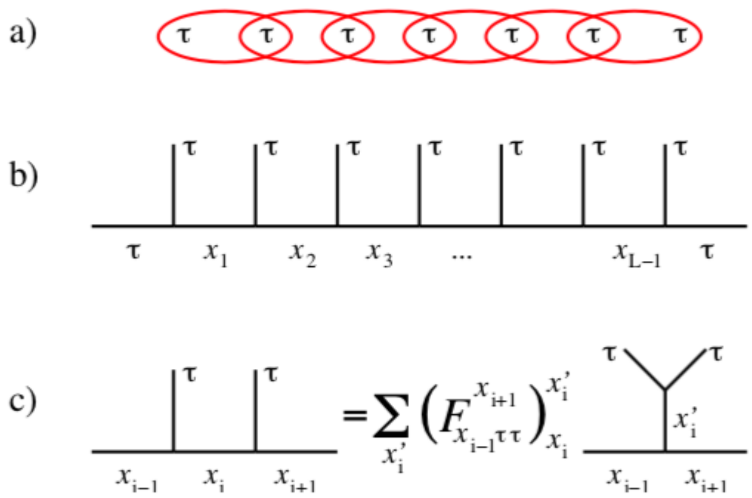
project out bulk



'anyon' chain model with
fusion category symmetry

Levin & Wen, PRB 2005; Feiguin, Trebst, et al PRL 2007

Fibonacci anyon chain

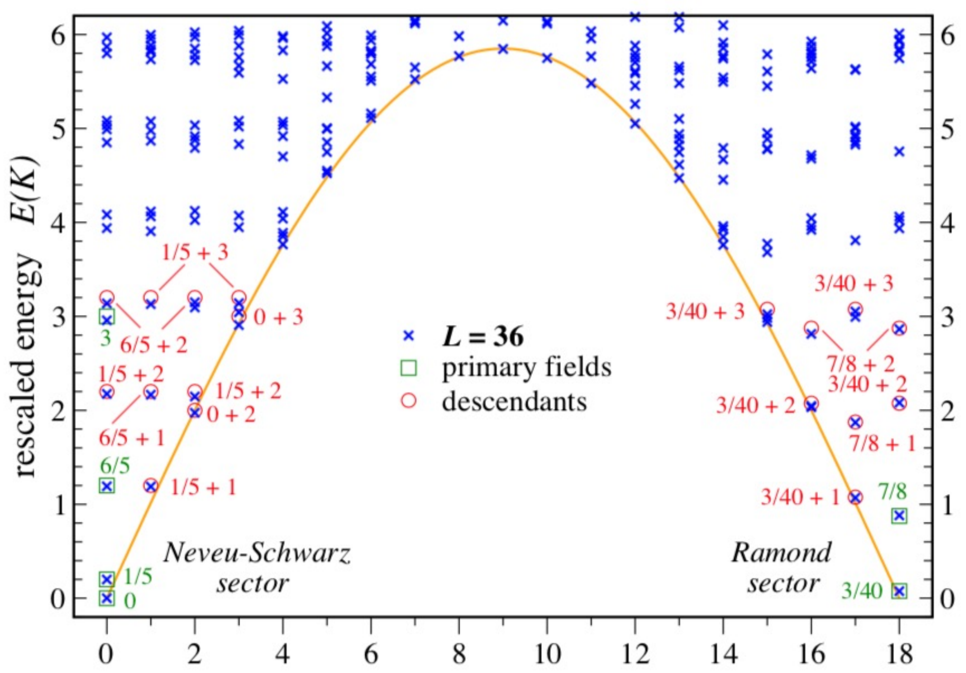


$$H = \sum_i H_i$$

$$\mathbf{H}_i|x_{i-1}x_ix_{i+1}\rangle = \sum_{x'_i=1,\tau} (\mathbf{H}_i)^{x'_i}_{x_i} \; |x_{i-1}x'_ix_{i+1}\rangle$$

$$\text{with} \quad (\mathbf{H}_i)^{x'_i}_{x_i} := -\left(F^{x_{i+1}}_{x_{i-1}\tau\tau}\right)^1_{x_i} \left(F^{x_{i+1}}_{x_{i-1}\tau\tau}\right)^1_{x'_i} .$$

Tri-critical Ising CFT



$$S_\tau \times S_\tau = 1 + S_\tau$$

Feiguin, Trebst, et al PRL 2007

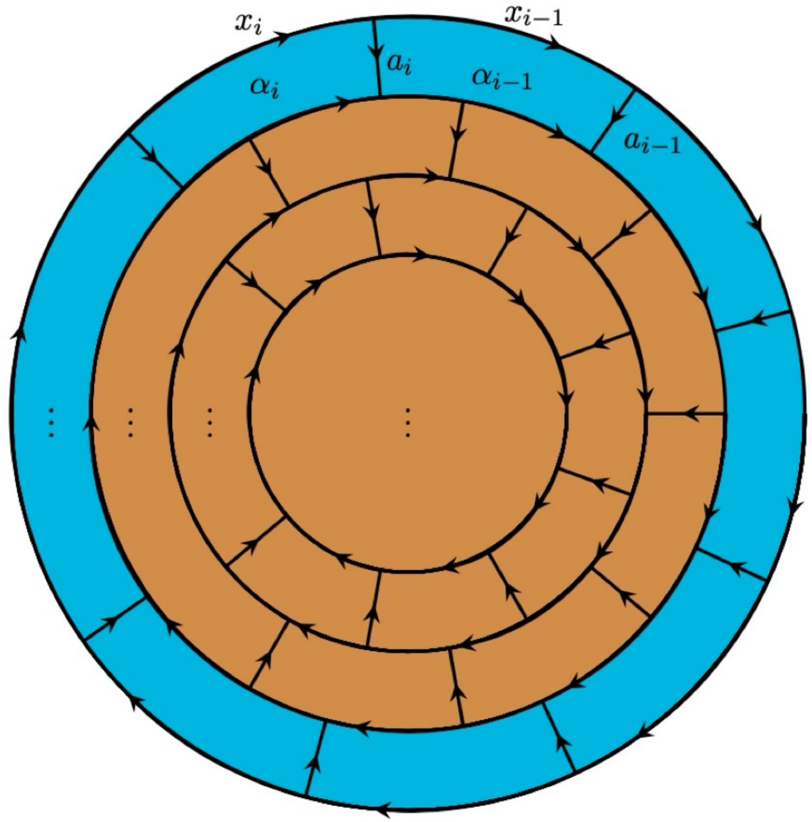
Generalization of anyon chain model

- ✦ Generalized symmetries can enhance the likelihood of gapless state or quantum criticality. Like conventional symmetries, they are **invariant** under renormalization group flow.

- ✦ Our goal is to "interpolate" edge theories of **symmetry-protected topological** phases (e.g., topological insulators) and **anyon chains**.

- ✦ To do that, we use **G-graded fusion category symmetry** to generalize the anyon-chain construction. In other words, we consider the boundary of symmetry-enriched topological phases of conventional group symmetry G.

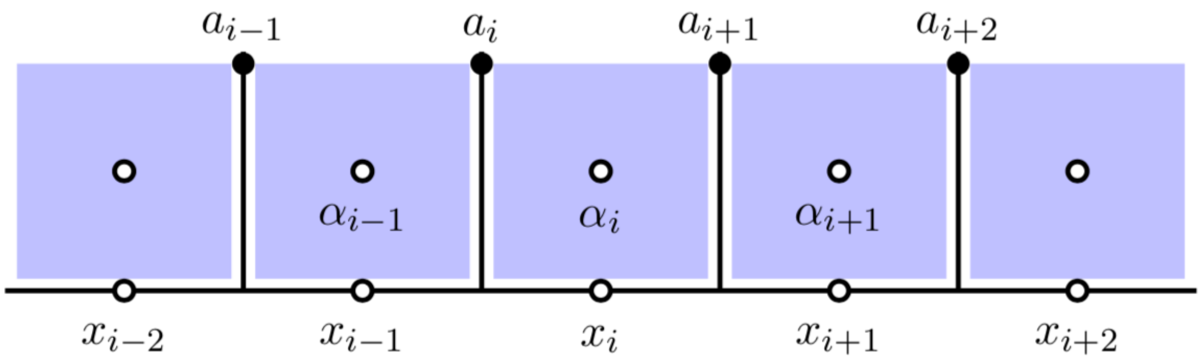
Edge of 2D symmetry-enriched topological phase



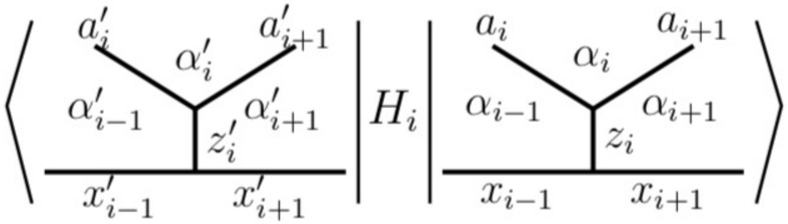
Low-energy edge Hilbert space is 1D-extensive

G-graded fusion category $\mathcal{C}_G = \mathcal{C}_0 \oplus \mathcal{C}_{g_1} \oplus \mathcal{C}_{g_2} \oplus \cdots$

Model with G-graded fusion category symmetry



$$\mathcal{H} = \bigoplus_{\{\alpha_i\}} \mathcal{H}^{\text{fusion}}_{\{\alpha_i\}}$$



$$= w^{z_i}_{\alpha_i^{-1}\alpha_i'} \delta^{\alpha_{i-1}'}_{\alpha_{i-1}} \delta^{\alpha_{i+1}'}_{\alpha_{i+1}} \delta^{x_{i-1}'}_{x_{i-1}} \delta^{x_{i+1}'}_{x_{i+1}} \delta^{z_i'}_{z_i}$$

Heinrich-Burnell-Fidkowski-Levin PRB 94, 235136 (2016); Cheng-Gu-Jiang-Qi, PRB 96 115107 (2017)

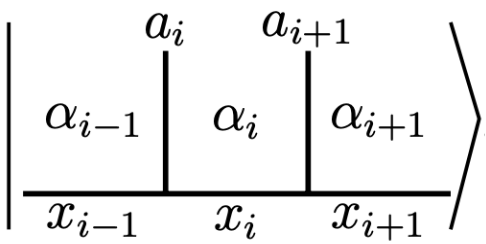
Examples

- ✦ G trivial, $\mathcal{C}_G = \mathcal{C}_0$ => anyon chain
- ✦ $\mathcal{C}_0$ trivial, $\mathcal{C}_G = (G, \nu_3)$ => edge of 2D SPT
- ✦ $\mathbb{Z}_N$ Tambara-Yamagami category

$$G = \mathbb{Z}_2 = \{0,1\}$$

$$\mathcal{C}_{Z_N-TY} = \{1, \psi, \cdots, \psi^{N-1}\} \oplus \{\sigma\}$$

$$|x_{i-1}x_i x_{i+1}\rangle \equiv$$



$$H = -\sum_i H_i \qquad \kappa = \pm 1$$

$$H_i|\mu\mu\mu\rangle = \cos\theta|\mu\sigma\mu\rangle$$

$$H_i|\mu\mu\sigma\rangle = r|\mu\mu\sigma\rangle + \sin\theta|\mu\sigma\sigma\rangle$$

$$H_i|\mu\sigma\nu\rangle = \delta_{\mu\nu}\cos\theta|\mu\mu\mu\rangle$$

$$H_i|\sigma\mu\mu\rangle = r|\sigma\mu\mu\rangle + \sin\theta|\sigma\sigma\mu\rangle$$

$$H_i|\mu\sigma\sigma\rangle = r|\mu\sigma\sigma\rangle + \sin\theta|\mu\mu\sigma\rangle$$

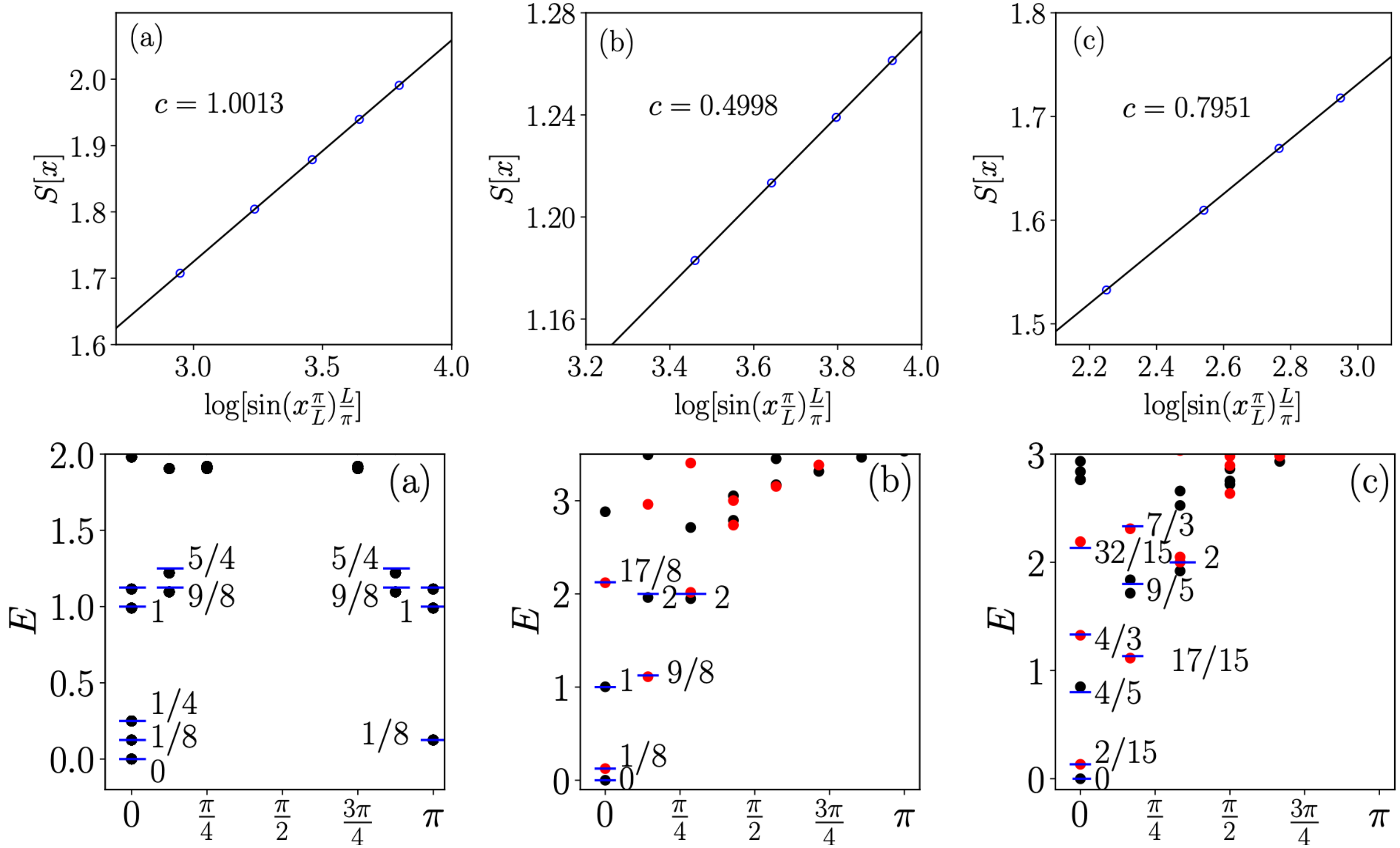
$$H_i|\sigma\mu\sigma\rangle = \frac{\kappa\cos\theta}{\sqrt{|A|}}|\sigma\sigma\sigma\rangle$$

$$H_i|\sigma\sigma\mu\rangle = r|\sigma\sigma\mu\rangle + \sin\theta|\sigma\mu\mu\rangle$$

$$H_i|\sigma\sigma\sigma\rangle = \frac{\kappa\cos\theta}{\sqrt{|A|}}\sum_{\mu\in A}|\sigma\mu\sigma\rangle$$

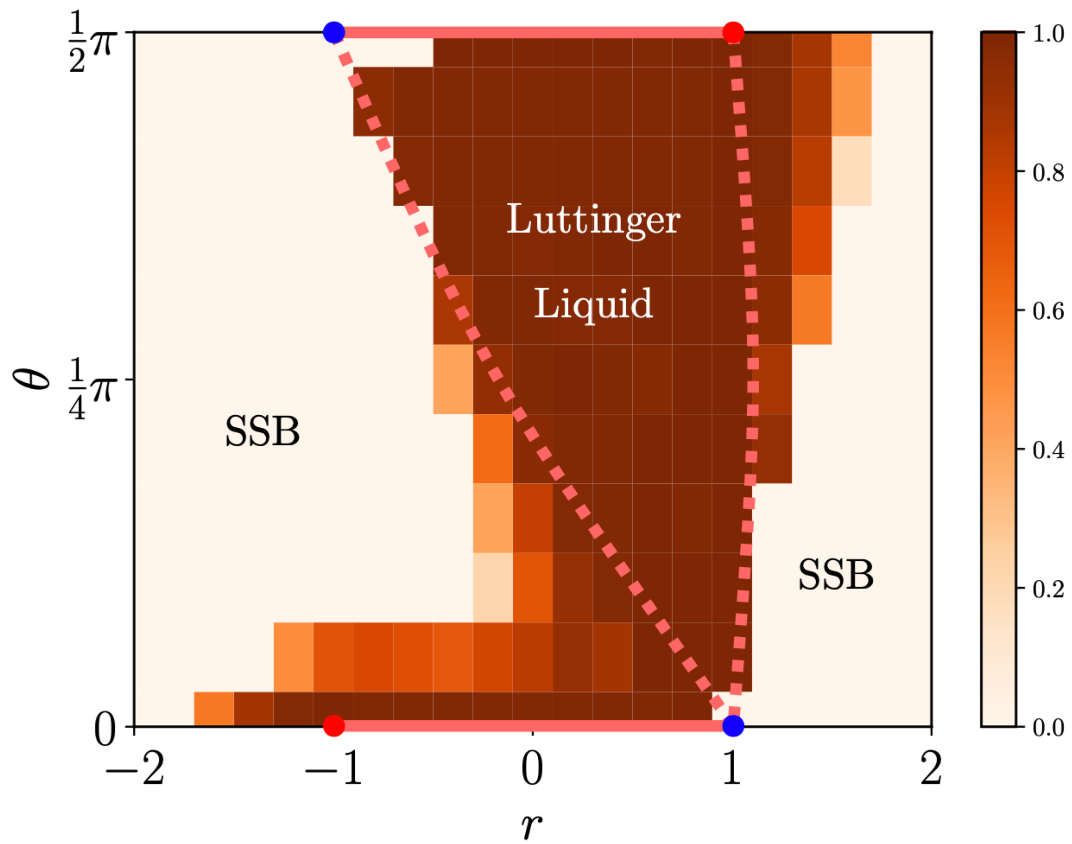
Central charge and spectrum of $\mathbb{Z}_N$ Tambara-Yamagami model

N=1　　N=2　　N=3



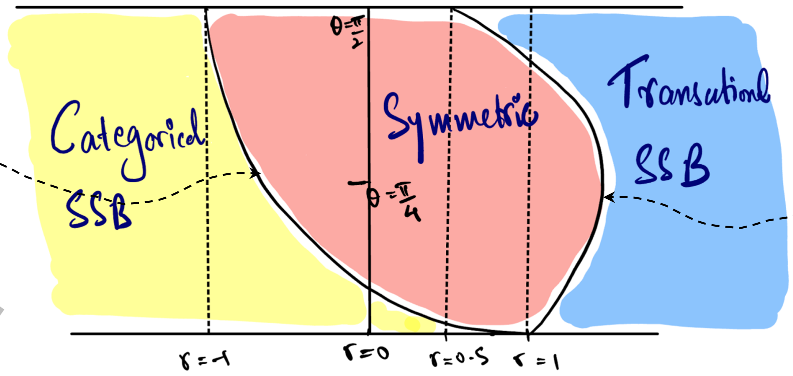
$r = 0, \theta = \pi/4$　　$r = 0, \theta = \pi/4$

Phase diagram of N=1 model
(Z_2 SPT edge)



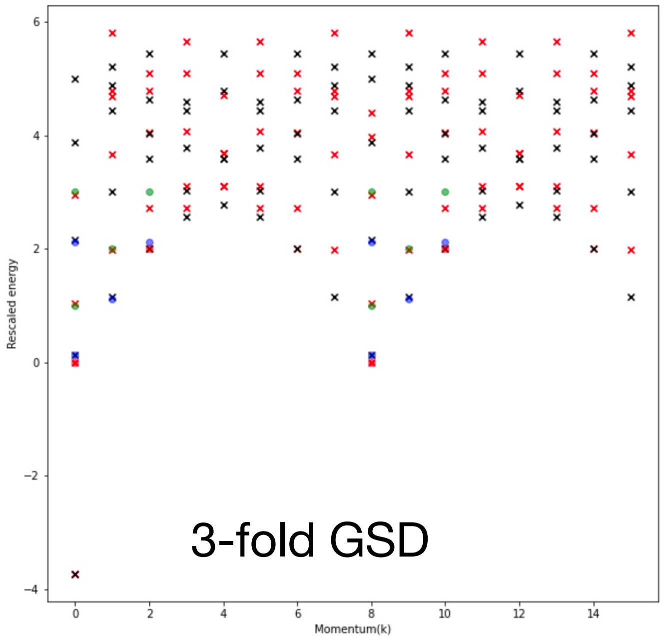
Color: central charge from DMRG

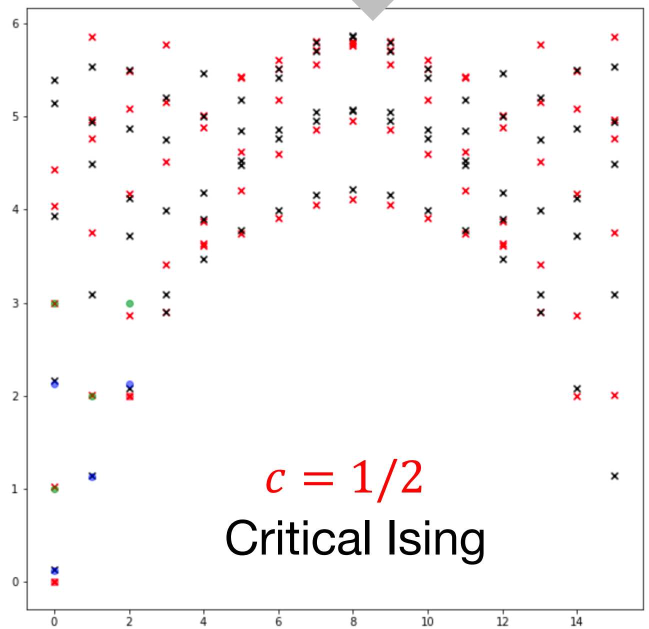
Skematic Phase diagram of model with C_{Ising} Symmetry

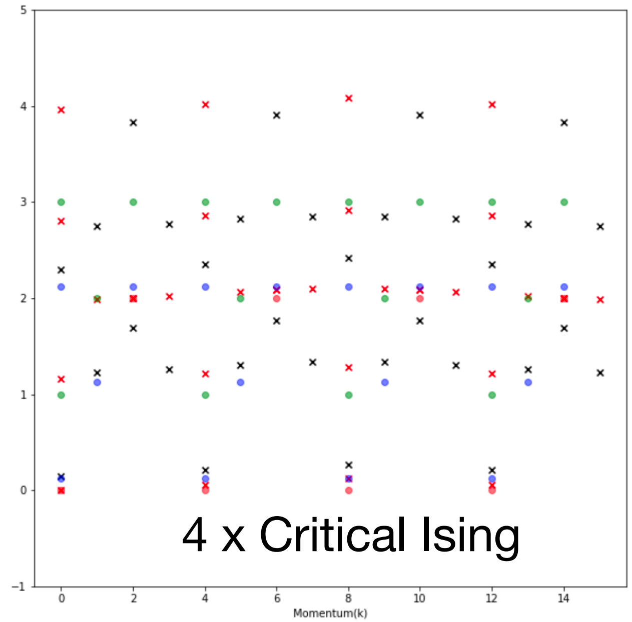


Gapped SSB to critical Ising: Tri-critical Ising

Critical Ising to gapless SSB: c~1.5







Outlook

- ✦ Edge models of fermionic SPT/SETs (properly encode the Z2-grading structure)

- ✦ Understand transition between critical phase and gapless symmetry breaking phase?

- ✦ Generalization to 2+1D dimensions?

- ✦ Simulation in cold atomic systems?

国家自然科学基金委员会
N S F C

Research Grants Council 研究資助局