100 Years of Quantum Physics, ICISE Quy Nhon (2025)

General Properties of Frustration-Free Quantum Many-Body Systems

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Bosons/Spins

Lee-Katsura-HW, arXiv:2310.16881 (PRL 2024)
Masaoka-Soejima-HW, arXiv:2406.06414 (PRB 2025)
Masaoka-Soejima-HW, arXiv:2406.06415
Masaoka-Soejima-HW, arXiv:2502.09908 (J. Stat. Phys. 2025)

Fermions

Masaoka-Ono-Po-HW, arXiv:2503.12879
Ono-Masaoka-HW-Po, arXiv:2503.14312
Sengoku-Po-HW, arXiv:2505.01010 (PRB 2025)

Introduction & Motivation:

Two interesting models
I recently found…

1. U(1) Symmetry Breaking in 1D

HW-Katsura-Lee, PRL (2024) Editors' Suggestions

- $s = 1$ XXZ spin chain with four-spin interaction ($\Delta \neq 1$).

$$\hat{H}_i = -J(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z) + \frac{J}{\Delta}\left[1-(1-\Delta)(\hat{s}_i^z)^2\right]\left[1-(1-\Delta)(\hat{s}_{i+1}^z)^2\right]$$

$G = \mathrm{SO}(2)$: Spin rotation symmetry about z axis generated by $\hat{S}^z = \sum_{i=1}^{L} \hat{s}_i^z$.

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- Order parameter $\hat{\mathcal{O}} = \sum_{i=1}^{L} \hat{s}_i^x$. $[\hat{H}, \hat{\mathcal{O}}] \neq 0$ when $\Delta \neq 1$.

- Symmetry-breaking field: $\hat{H}(h) = \hat{H} - h\hat{\mathcal{O}}$.

- $m(h) = \dfrac{\langle \hat{\mathcal{O}} \rangle}{V}$ for the ground state of $\hat{H}(h)$.

- Spontaneous symmetry breaking

  $\Leftrightarrow \quad \lim_{h \to +0} \lim_{V \to \infty} m(h) \neq 0$

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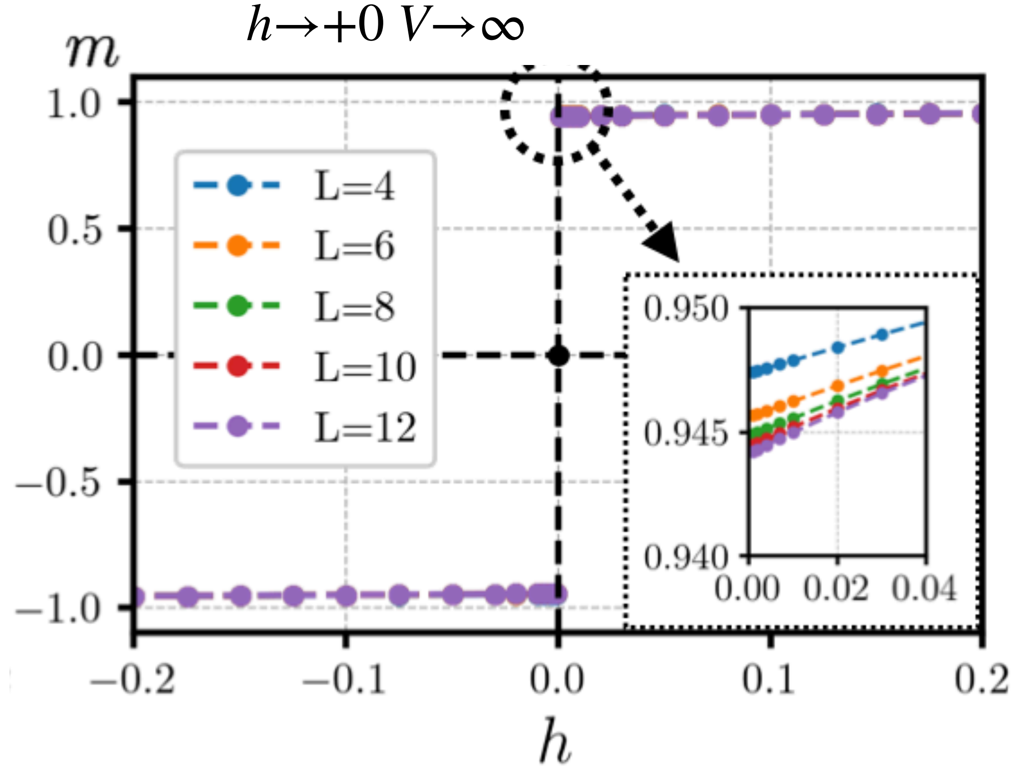

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Somehow the spin-wave excitation of this model has quadratic dispersion $\omega_{\vec{k}} \propto k^2$ (like in Heisenberg ferromagnet)

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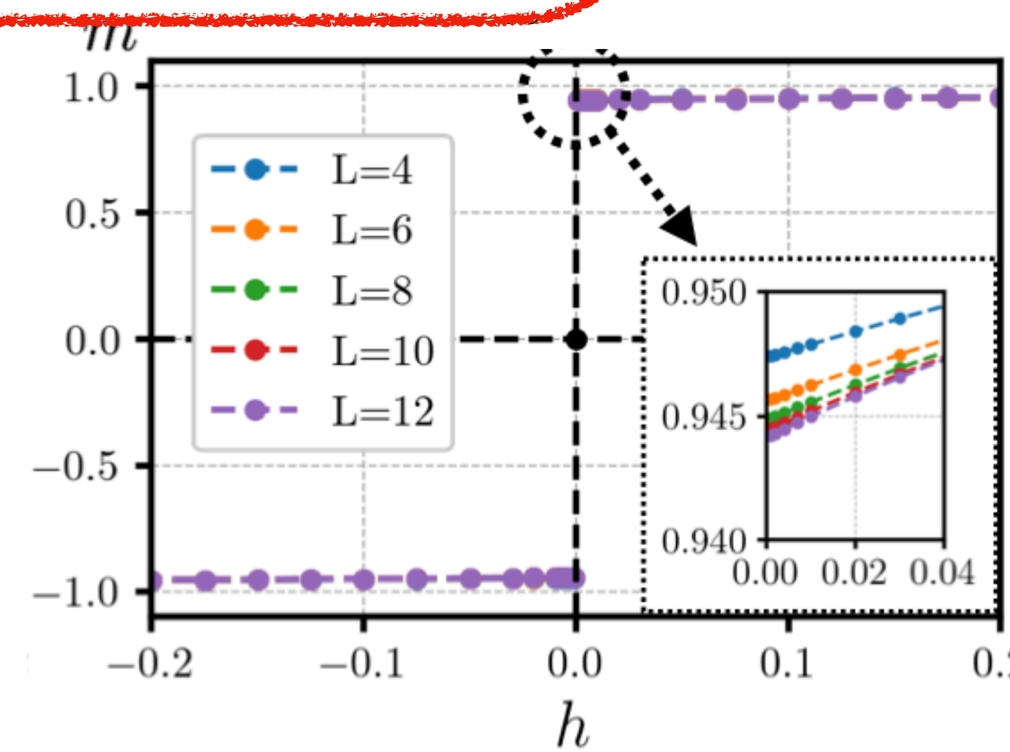

Hohenberg-Mermin-Wagner theorem at $T > 0$

- Hohenberg-Mermin-Wagner (HMW) theorem:

Hohenberg (1967), Mermin-Wagner (1966)

Continuous symmetries cannot be broken at finite T in $d \leq 2$.

spatial dimension (not including time)

Hohenberg-Mermin-Wagner theorem at $T > 0$

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- Nambu-Goldstone (NG) theorem: Nambu (1960), Goldstone (1961)

Spontaneously broken continuous symmetry $\Rightarrow$ Gapless excitations

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- Proof of HMW theorem (by contradiction)
  1. Suppose a continuous symmetry is broken.
  2. NG theorem implies gapless excitations (Nambu-Goldstone bosons).
  3. Infrared divergence originating from NG bosons in $d \leq 2$ destroys the order parameter.

Hohenberg-Mermin-Wagner theorem at $T = 0$

- Hohenberg-Mermin-Wagner (HMW) "theorem" at $T = 0$: Hohenberg (1967), Mermin-Wagner (1966)

Continuous symmetries cannot be broken at ~~finite T in $d \leq 2$~~.
$T = 0$ in $d \leq 1$

Coleman (1973)
※Lorentz symmetry assumed

- Nambu-Goldstone (NG) theorem: Nambu (1960), Goldstone (1961)

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Proof of HMW theorem for a finite T via Bogoliubov inequality

Hohenberg (1967), Mermin-Wagner (1966)

- Fourier transformation: $\hat{Q}_{\vec{k}} = \sum_{\vec{r}\in\Lambda} \hat{Q}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$, $\hat{X}_{\vec{k}} = \sum_{\vec{r}\in\Lambda} \hat{X}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$ $(\hat{\mathcal{O}} = [i\hat{Q}, \hat{X}])$

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$$\frac{1}{V^2}\sum_{\vec{k}} \langle \hat{X}_{\vec{k}}\hat{X}^\dagger_{\vec{k}} + \hat{X}^\dagger_{\vec{k}}\hat{X}_{\vec{k}}\rangle \quad \geq \quad \frac{1}{V}\sum_{\vec{k}} \frac{2T \ \ |\frac{1}{V}\langle [i\hat{Q}^\dagger_{\vec{k}}, \hat{X}_{\vec{k}}]\rangle\,|^2}{\frac{1}{V}\langle [[\hat{Q}_{\vec{k}}, \hat{H}], \hat{Q}^\dagger_{\vec{k}}]\rangle}$$

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$O(1)$ $\qquad\qquad$ $\simeq Ck^2$

$$O(1) \geq \int \frac{d^d k}{(2\pi)^d} \frac{2T\,|m|^2}{Ck^2}$$

IR divergence in $d \leq 2$ $\;\Rightarrow\;$ $m = 0$ (i.e., no SSB)

Proof of HMW theorem for $T = 0$ via Bogoliubov inequality

Takada (1975)

- Fourier transformation: $\hat{Q}_{\vec{k}} = \sum_{\vec{r}\in\Lambda} \hat{Q}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$, $\hat{X}_{\vec{k}} = \sum_{\vec{r}\in\Lambda} \hat{X}_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}$ $(\hat{\mathcal{O}} = [i\hat{Q}, \hat{X}])$

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$$\frac{\langle\hat{\mathcal{O}}\rangle}{V} = m$$

In our example $n = 2$.
SSB is allowed
in $d > 0$.

$\omega_{\vec{k}} \simeq v k^n$

$$O(1) \geq \int \frac{d^d k}{(2\pi)^d} \frac{\cancel{2T}\,|m|^2}{Ck^2}$$

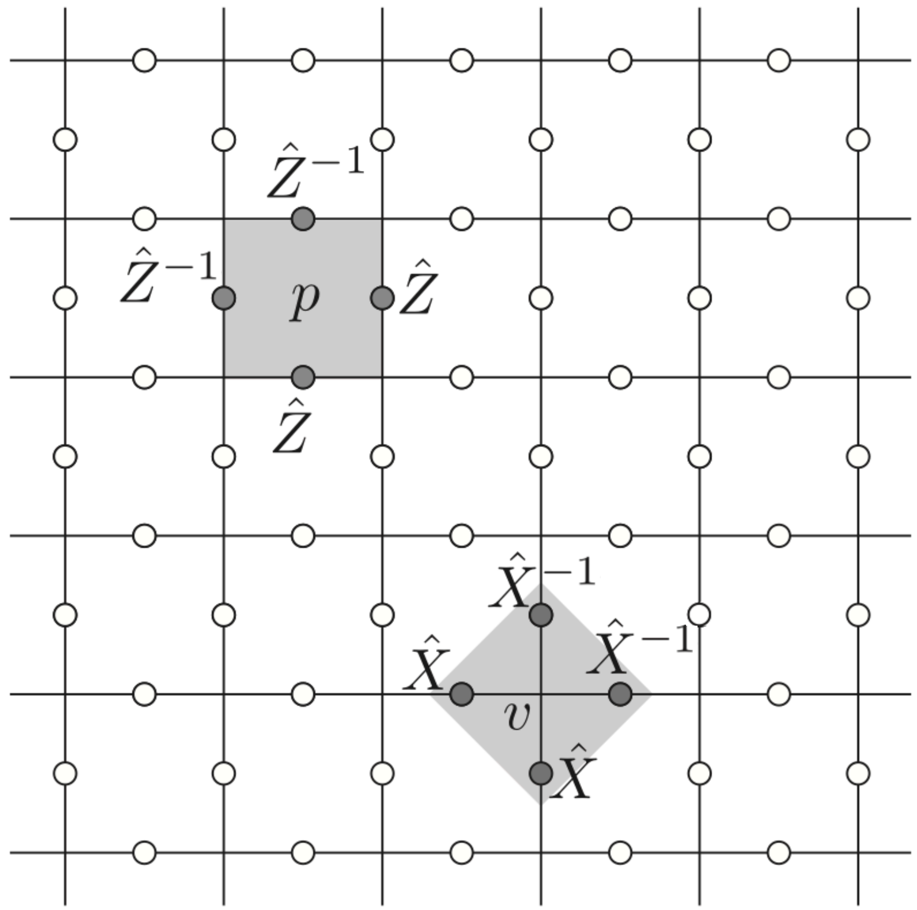
$d \leq 2 - n$

IR divergence in $\cancel{d \leq 2}$ $\Rightarrow$ $m = 0$ (i.e., no SSB)

2. Modified $\mathbb{Z}_N$ Toric Code

HW-Cheng-Fuji (2023)

Standard $\mathbb{Z}_N$ toric code



$$X = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}$$

$$ZX = \omega XZ = \begin{pmatrix} & & & & 1 \\ \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{N-1} & \end{pmatrix} \qquad Z^N = X^N = 1$$

- Topological degeneracy: $N_{\text{deg}} = N^2$ under PBC
- Topological entanglement entropy: $S_{\text{top}} = -\log N \neq 0$
- Anyonic excitations

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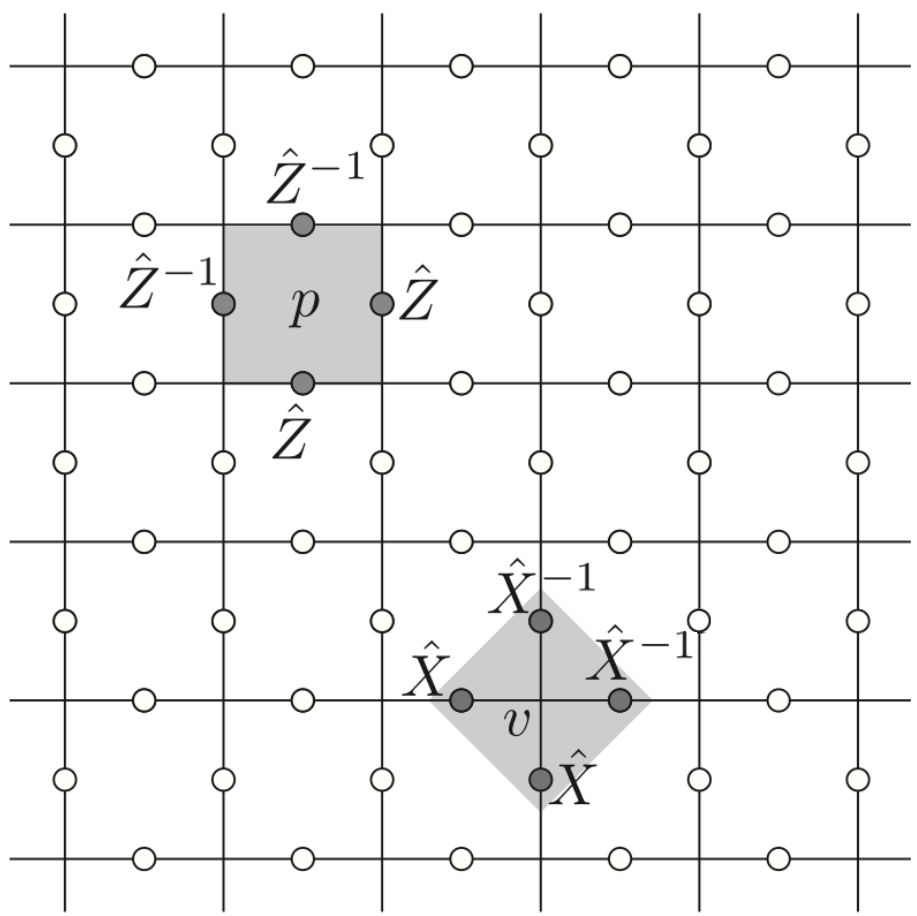
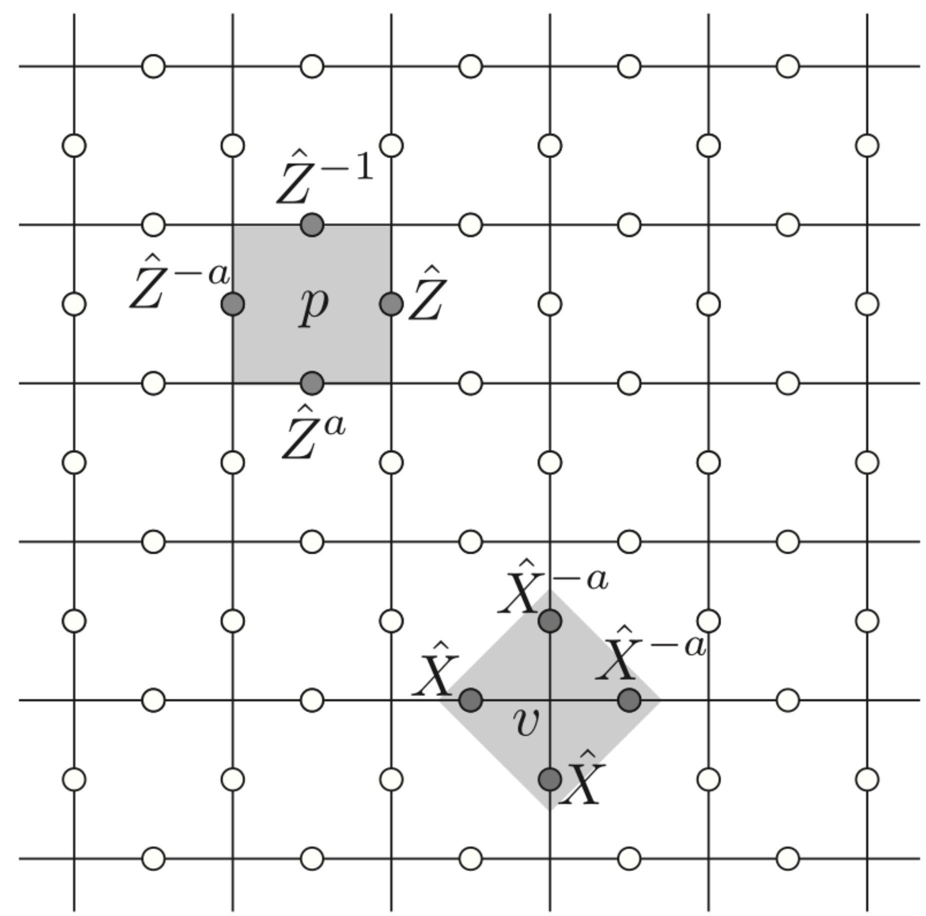


Integer parameter a

Modified $\mathbb{Z}_N$ toric code



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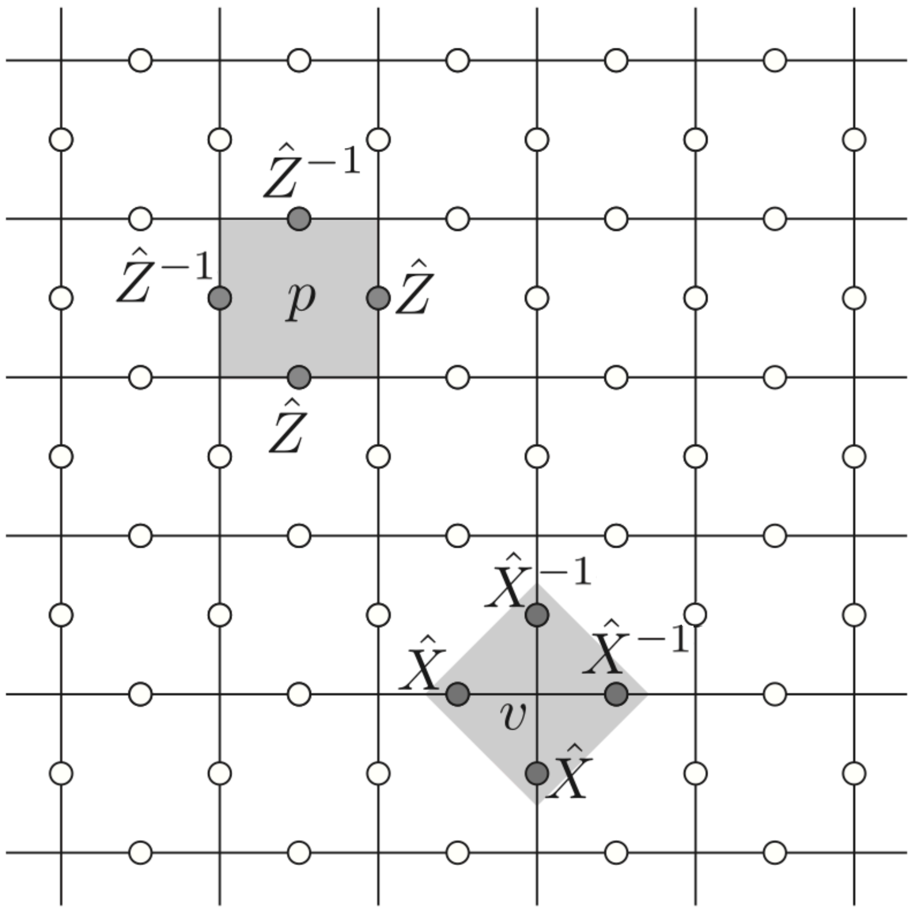

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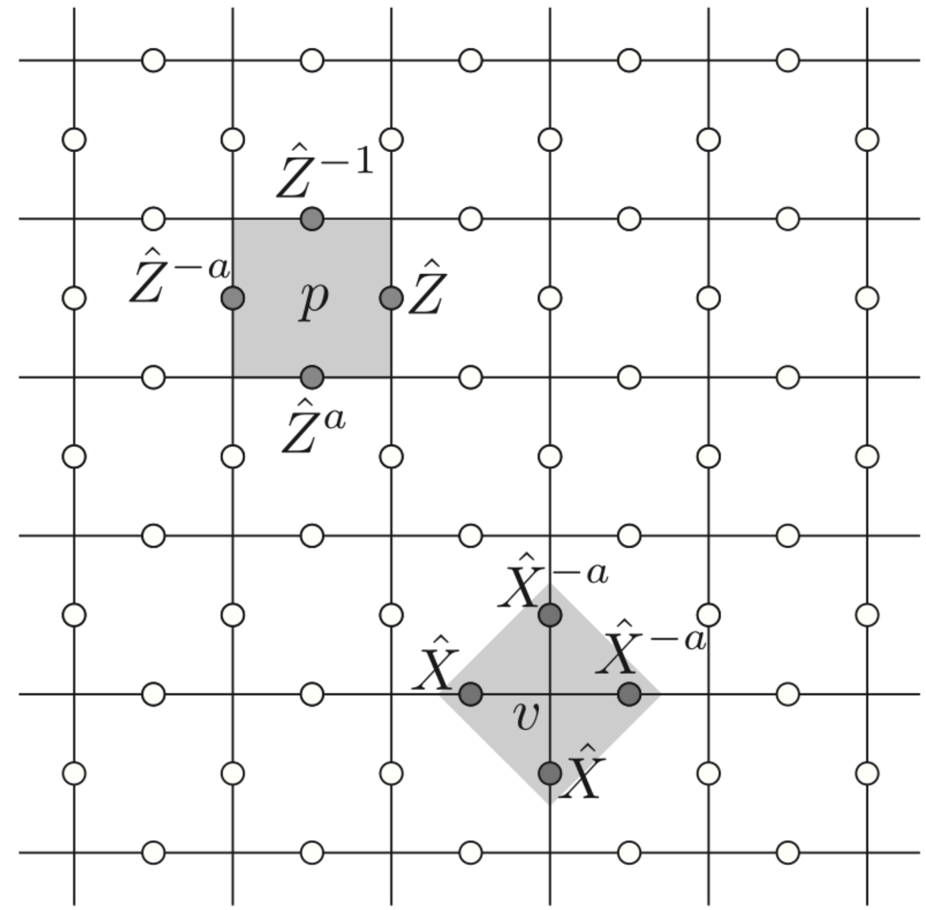


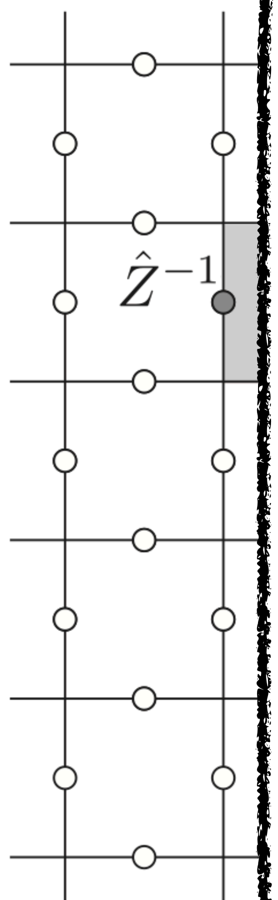
- $N_{\mathrm{deg}} = N^2$ under PBC
- Topologically-ordered $S_{\mathrm{top}} = -\log N \neq 0$

- $N_{\mathrm{deg}} = [\gcd(a^{L_1} - 1, a^{L_2} - 1, N)]^2$ → can be 1!
  e.g. $N = 11$, $a = 2$, $L_{1,2} \neq 0$ mod 10
- $S_{\mathrm{top}} \neq 0$ unless a is a multiple of $\mathrm{rad}(N)$

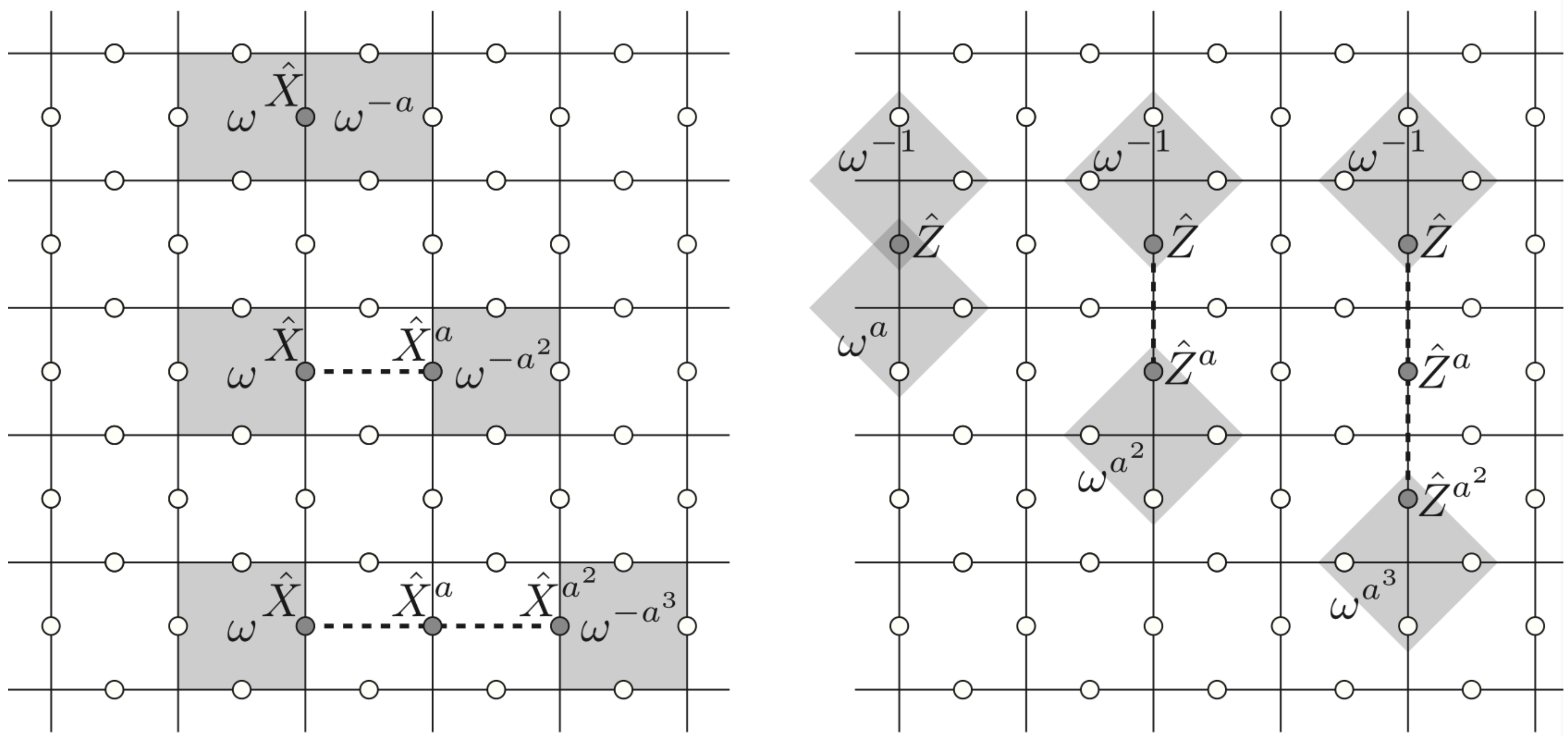
2. Modified $\mathbb{Z}_N$ Toric Code

HW-Cheng-Fuji (2023)

Standa



Excitations can be moved by modulated strings



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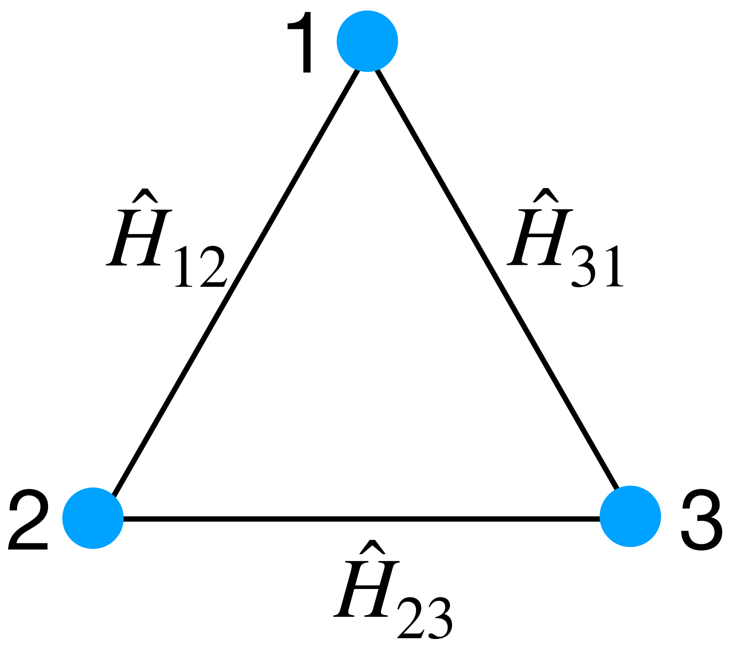
These models are frustration free!!
→ Explore general properties
of FF systems

Frustration in Quantum Many-Body Systems

- Antiferromagetic interaction among three spins 1,2,3.
  $\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}$



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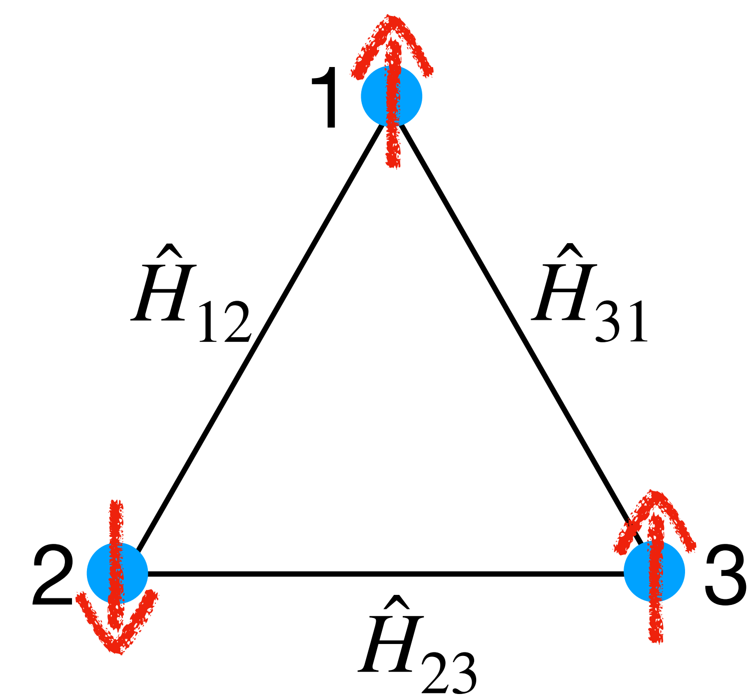

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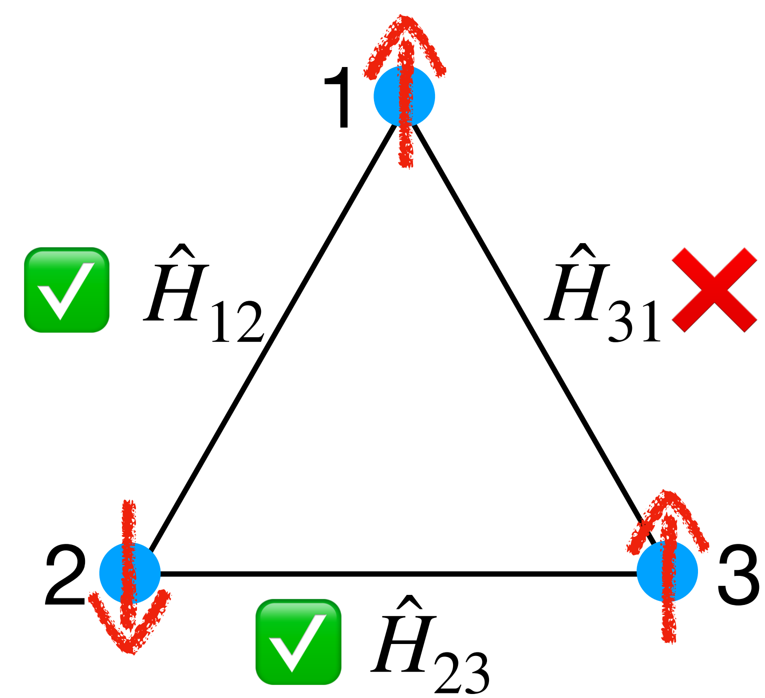

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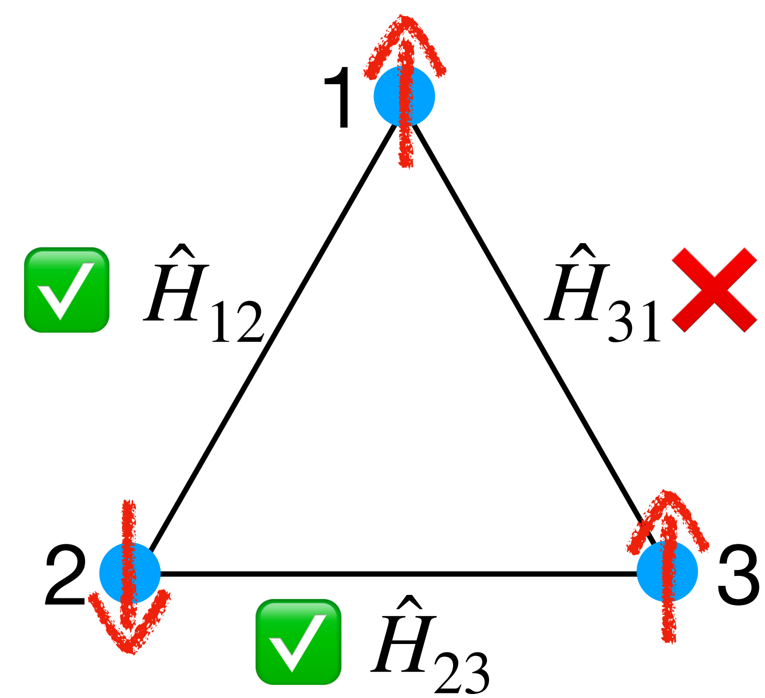
  → *frustration*.



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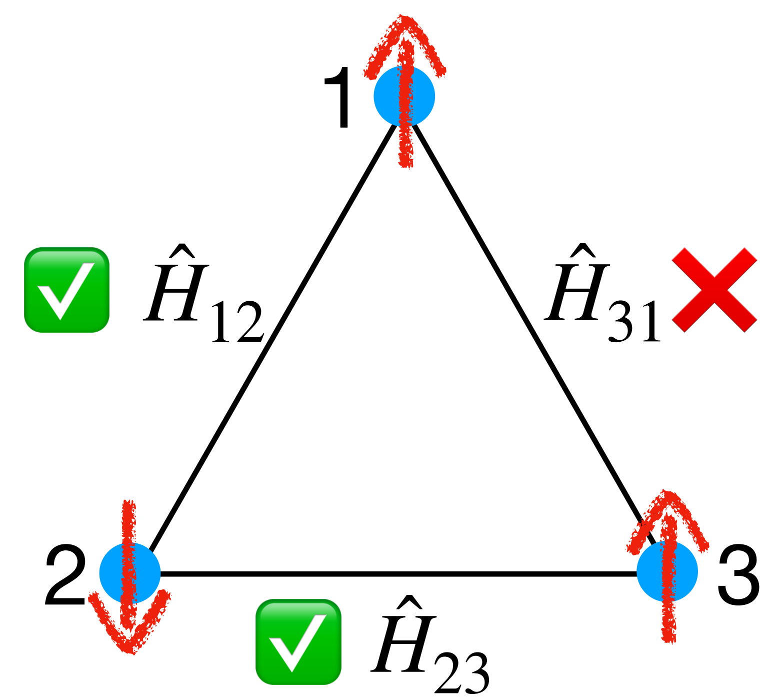
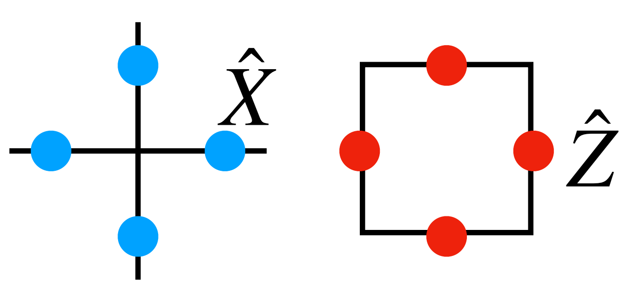


- More generally, $\hat{H} = \sum_i \hat{H}_i$ is *frustration free* if
  - $\hat{H}_i$'s are finite ranged.
  - Ground state $|\Phi_{\mathrm{GS}}\rangle$ of $\hat{H}$ minimizes all $\hat{H}_i$ simultaneously.
    i.e., $\hat{H}_i|\Phi_{\mathrm{GS}}\rangle = E_{\mathrm{GS},i}|\Phi_{\mathrm{GS}}\rangle$ and $E_{\mathrm{GS},i}$ is GS energy of $\hat{H}_i$.

$\hat{H}_i$'s *do not* have to commute with each other.

Examples of FF spin models

- Paramagnet:
  $$\hat{H}_i = -\hat{s}_i^z$$
  Trivial
- Majumdar-Ghosh model:
  $$\hat{H}_i^{(S=1/2)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \hat{\vec{s}}_{i+1} \cdot \hat{\vec{s}}_{i+2} + \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+2}$$
  SSB of translation
- AKLT:
  $$\hat{H}_i^{(S=1)} = \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1} + \frac{1}{3}(\hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1})^2$$
  SPT
- Toric code:
  $$\hat{H} = -\sum_{+} \hat{V}_{+} - \sum_{\square} \hat{P}_{\square}$$
  

  Topological Order
- Other commuting projector Hamiltonians

Goal

- We want to understand
  - General properties and limitations of FF Hamiltonians.

Symmetry

- Spontaneous symmetry breaking
- Topological phases

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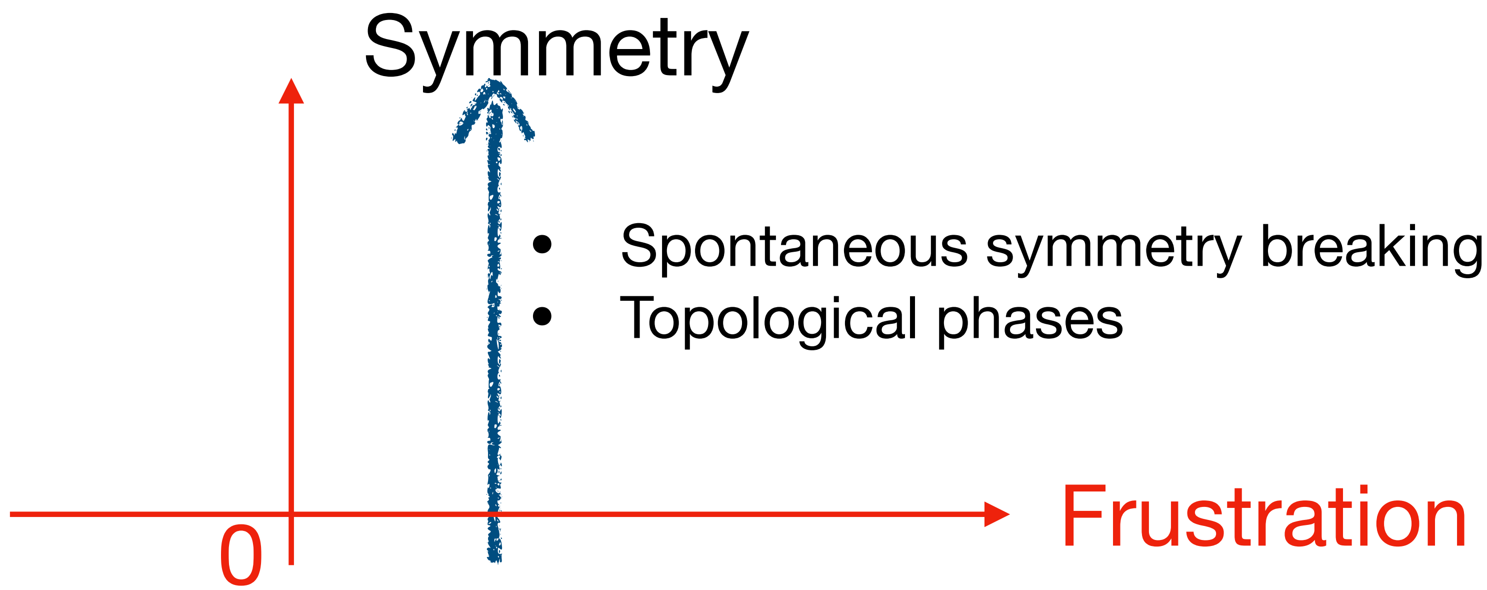

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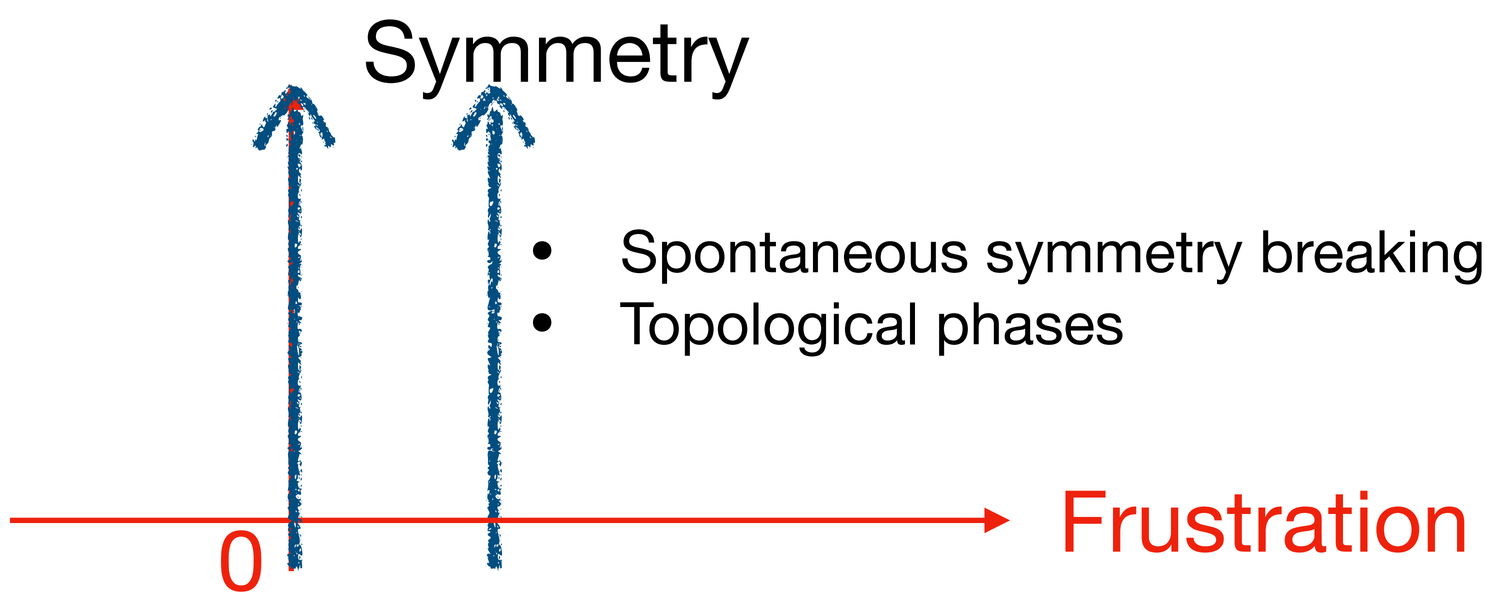

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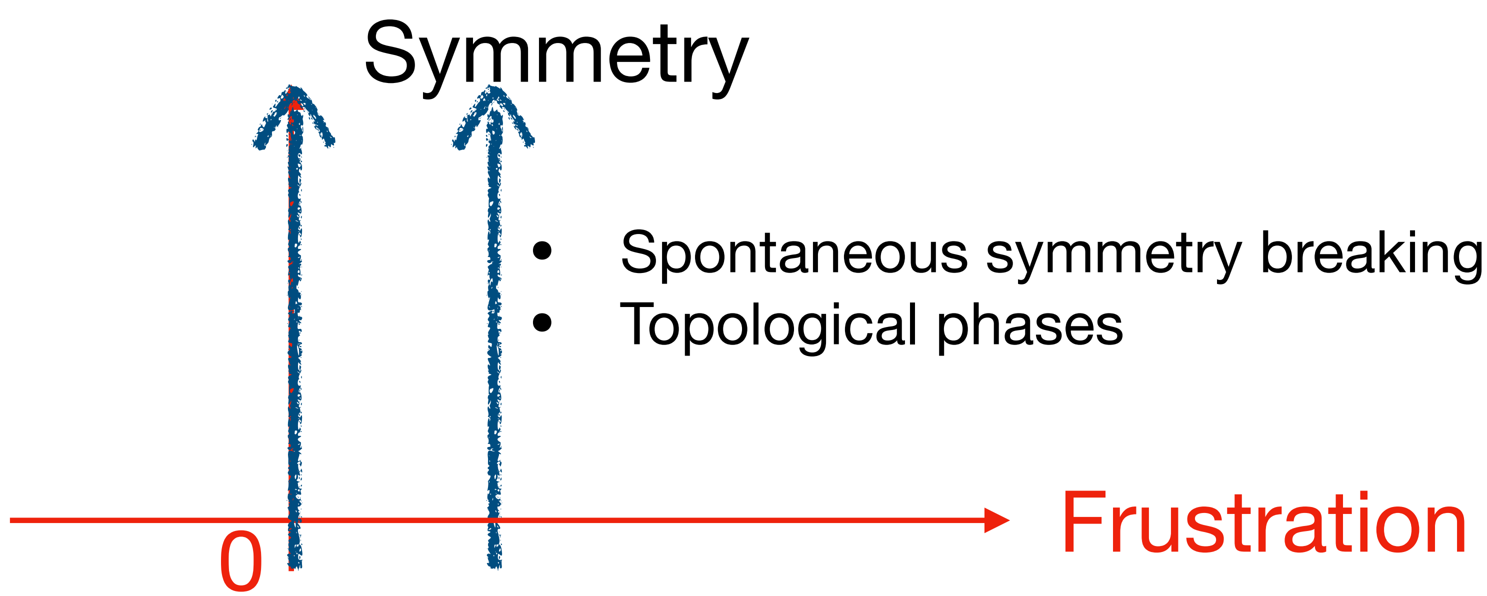
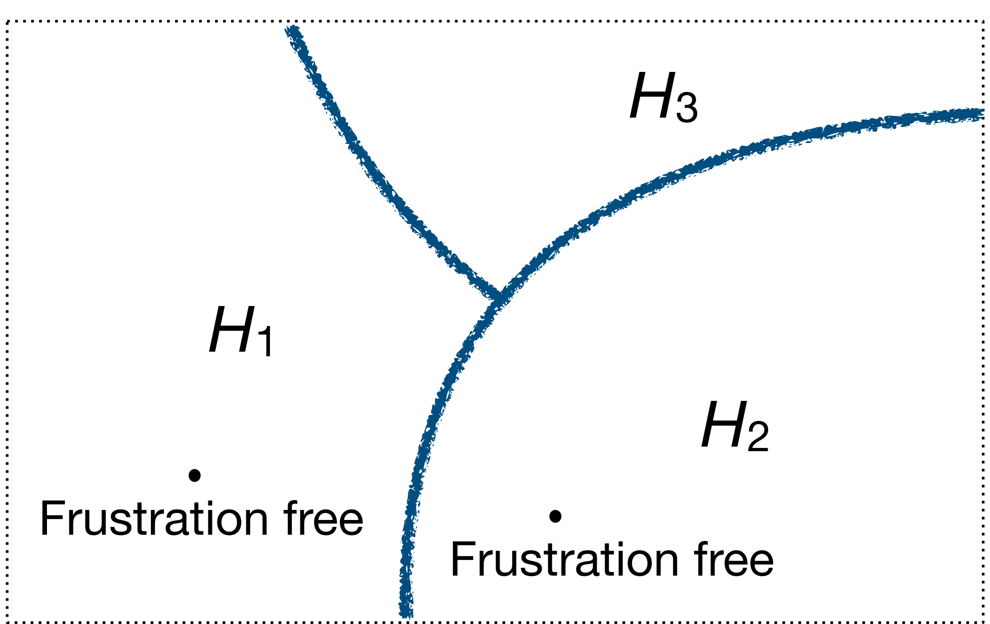
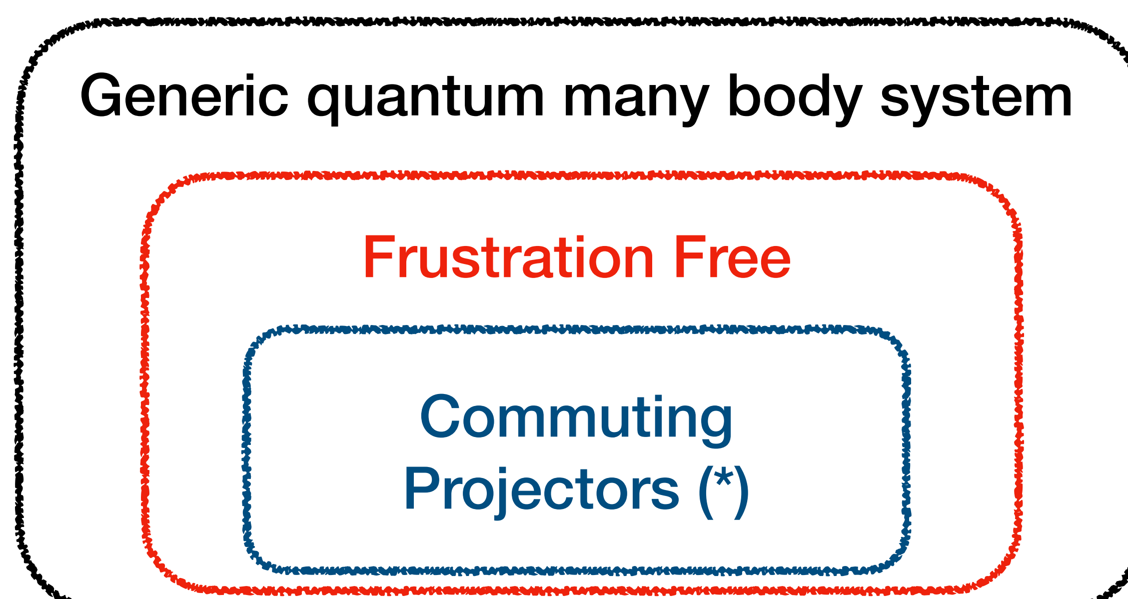


  - Which phases can be represented by FF Hamiltonians.





Good balance of difficulty and simplicity

Conjectures on FF systems

Masaoka-Soejima-HW (2024)

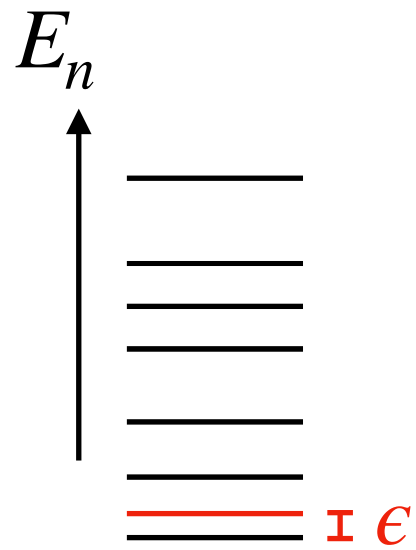
Sengoku-Po-HW (2025)

Scaling of finite-size gap

Masaoka-Soejima-HW PRB (2024)

Conjecture 1: If $\hat{H}$ is FF and gapless, then

the finite size gap of $\hat{H}$ is $\epsilon \sim L^{-z}$ with $z \geq 2$.



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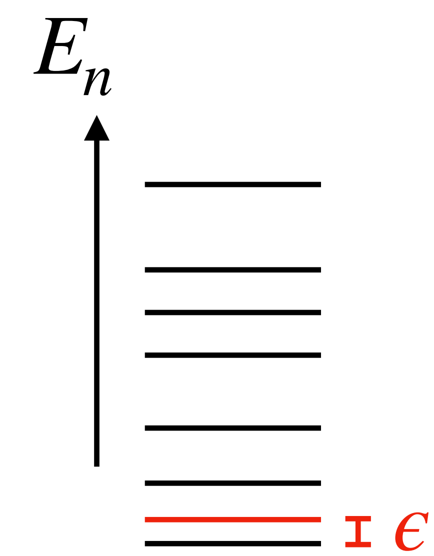

With translation symmetry, there exists $|\Psi_{\vec{k}}\rangle$ such that

- $\hat{T}_{\vec{a}}|\Psi_{\vec{k}}\rangle = e^{-i\vec{k}\cdot\vec{a}}|\Psi_{\vec{k}}\rangle$
- $\langle\Psi_{\vec{k}}|\hat{H}|\Psi_{\vec{k}}\rangle - E_0 = O(|\vec{k} - \vec{k}_0|^2)$

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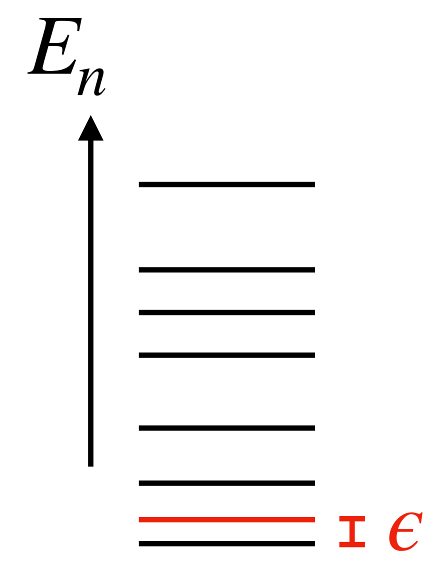

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Low-energy effective field theory of gapless FF system
cannot be Lorentz invariant.

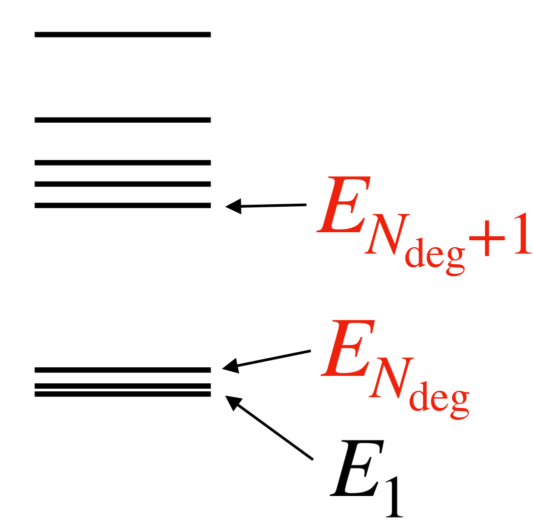
Absence of finite size splitting

Masaoka-Soejima-HW (2024)

- Arrange all eigenvalues of $\hat{H}$ as $E_1 \leq E_2 \leq \cdots \leq E_D$.

- In general (regardless of frustration),
  $\hat{H}$ is gapped $\Leftrightarrow$ there exists N_{deg} such that

$$\underbrace{\lim_{L\to\infty} E_{N_{\mathrm{deg}}} = E_1}_{\text{finite size splitting}} \text{ and } \underbrace{\lim_{L\to\infty} E_{N_{\mathrm{deg}}+1} \neq E_1}_{\text{energy gap}}.$$



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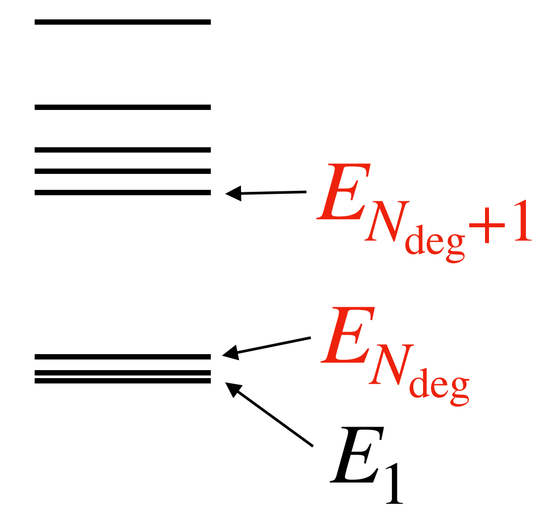
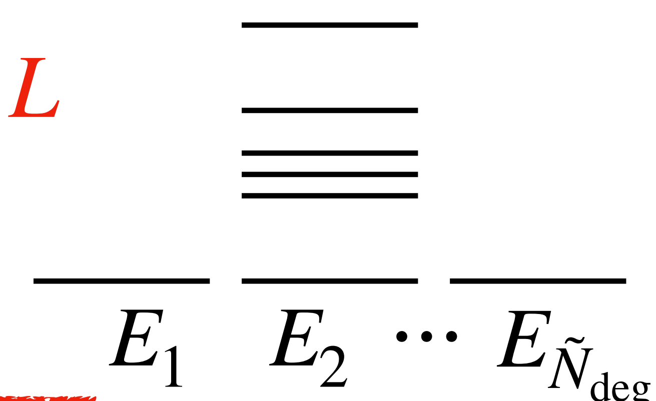
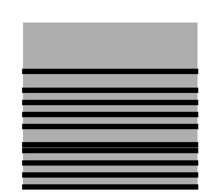


- Conjecture 2: If $\hat{H}$ is FF and gapped, $E_{N_{\mathrm{deg}}} = E_1$ for finite L

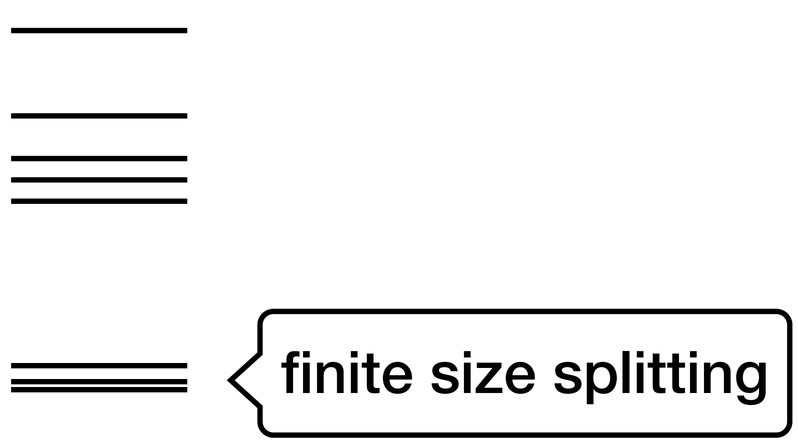


$$\lim_{L\to\infty} \epsilon = 0 \Leftrightarrow \hat{H} \text{ is gapless}$$



Examples

- Frustrated models
  - Transverse-field Ising model
  - perturbed MG model
  - Haldane model (OBC)
  - perturbed toric code (PBC)



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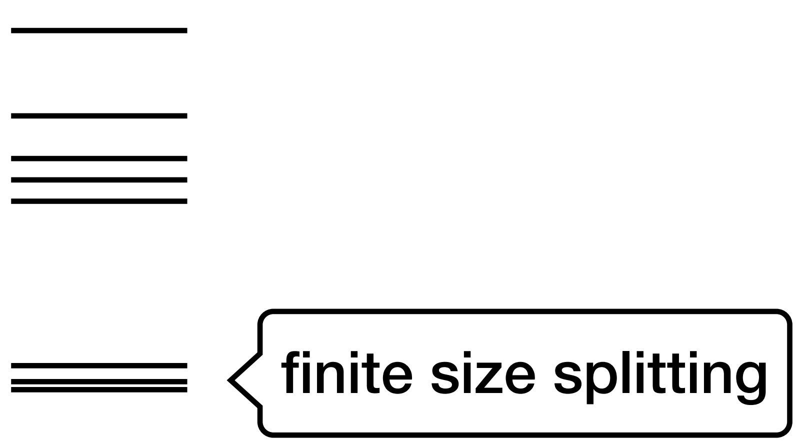

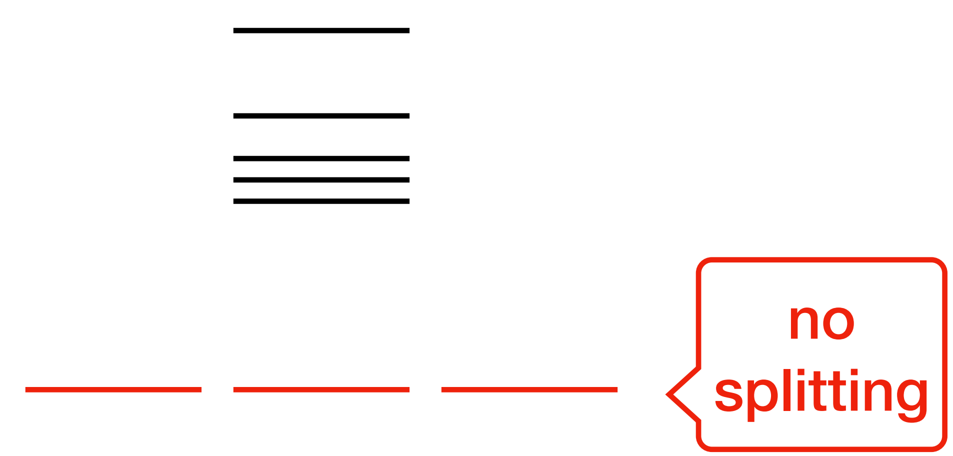

Absence of chiral phases

- No local commuting projector Hamiltonian
  - for thermal Hall states Kitaev (2006)
  - for electric Hall states Kapustin-Fidkowski (2020)
- Gaussian fermionic PEPS for Chern insulator has power-law correlations
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  Dubail-Read (2015)
- Conjecture 3: If $\hat{H}$ is FF, finite-ragened, and gapped,
  Hall conductance = 0.

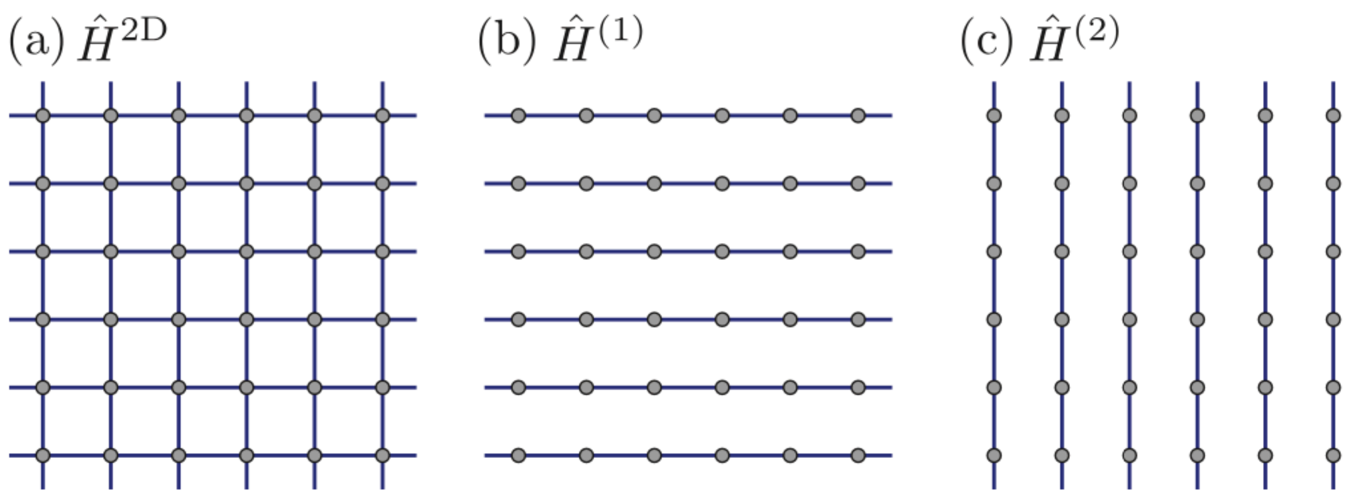
Ono-Masaoka-HW-Po, arXiv:2503.14312
Sengoku-Po-HW, arXiv:2505.01010

Partial Proof

S=1/2 spin models

- S=1/2
- Translation invariance
- Nearest-neighbor interactions
- d dimensional hypercubic lattice

Bravyi-Gosset (2015)
Masaoka-Soejima-HW (2024)



Only two gapless FF classes

Min-Max theorem: $\hat{H} \geq \hat{H}'$ implies $\lambda_n \geq \lambda_n'$

- $\mathrm{rank}[Q_{i,i+1}] = 1$:
  $\hat{Q}_{i,i+1} = |\psi\rangle\langle\psi|_{i,i+1}$ with
  $|\psi\rangle = (\alpha + i\beta)|0{,}1\rangle + (\alpha \pm i\beta)|1{,}0\rangle + \delta|1{,}1\rangle$
  ($\alpha = 0, \beta = 1/\sqrt{2}$ → Heisenberg model $\hat{Q}_{i,i+1} = \dfrac{1}{4} - \hat{\vec{s}}_i \cdot \hat{\vec{s}}_{i+1}$)
- $\mathrm{rank}[Q_{i,i+1}] = 2$:
  $\hat{Q}_{i,i+1} = \dfrac{1}{2} - \dfrac{1}{2}(e^{i\theta}\hat{s}_i^+\hat{s}_{i+1}^- + \mathrm{h.c.}) - \dfrac{1}{2}(\hat{s}_i^z + \hat{s}_{i+1}^z)$

New theorem

- Suppose $\hat{H}$ is frustration free.
- Also, suppose an equal-time correlation function shows a power-law decay (possible only in gapless system)

$$|\langle \Phi_{\mathrm{GS}}|\hat{\mathcal{O}}_{\vec{x}}^{\dagger}(\hat{1}-\hat{G})\hat{\mathcal{O}}'_{\vec{y}}|\Phi_{\mathrm{GS}}\rangle| \sim CL^{-p} \text{ for } |\vec{x}-\vec{y}| \sim L$$

$\hat{G}$ is the projector onto GS manifold.

Then, the finite size gap of $\hat{H}$ is

$$\epsilon = O((\log L)^2/L^2)$$

implying that $\epsilon \sim L^{-z}$ with $z \geq 2$.

Proof by Gosset-Huang inequality

- Hastings-Koma (2006): In general, in systems with spectral gap ϵ
  $|\langle\Phi_{\mathrm{GS}}|\hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1}-\hat{G})\hat{\mathcal{O}}'_{\vec{y}}|\Phi_{\mathrm{GS}}\rangle| \leq Ce^{-g'|\vec{x}-\vec{y}|\epsilon}$
  $\rightarrow$ Correlation length $\xi \sim \dfrac{1}{\epsilon}$

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- Gosset-Huang (2016): If H is frustration-free,
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- Consistent with $|\langle \Phi_{\mathrm{GS}}|\hat{\mathcal{O}}^{\dagger}_{\vec{x}}(\hat{1}-\hat{G})\hat{\mathcal{O}}'_{\vec{y}}|\Phi_{\mathrm{GS}}\rangle| \sim CL^{-p}$ $(|\vec{x}-\vec{y}| \sim L)$

  only when $\epsilon = O(L^{-2})$.

Unexpected application

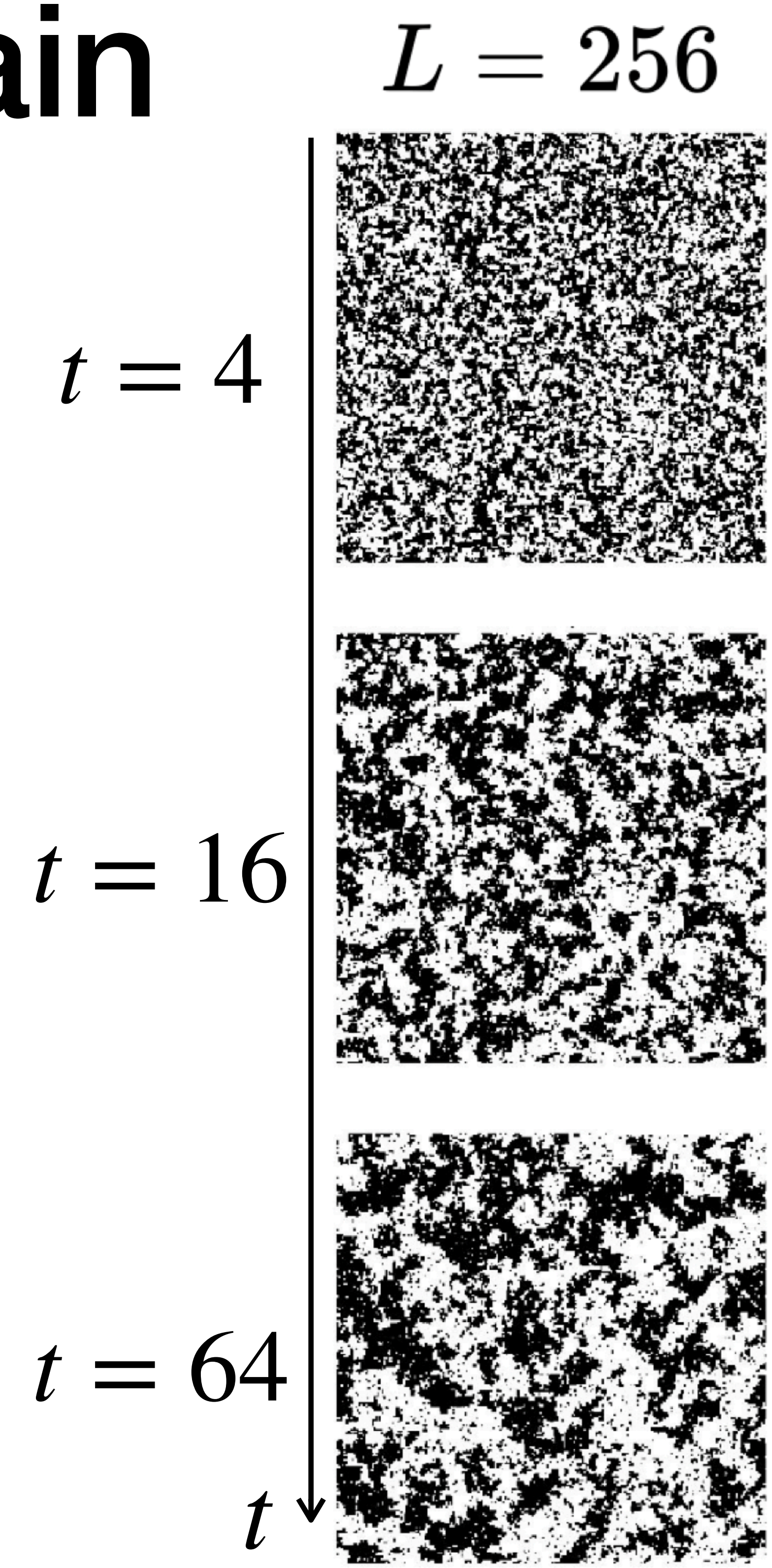
Relaxation to equilibrium state in Markov chain

- Boltzmann weight $w(C) = e^{-E(C)}$
  e.g. Ising model $E(C) = -J \sum_{(i,j)} \sigma_i \sigma_j$

- Master equation:
  $$\frac{d}{dt} p(t, C) = \sum_{C' \in S} W_{C,C'} p(t, C')$$

- Local update rule: $W = \sum_i W_i$

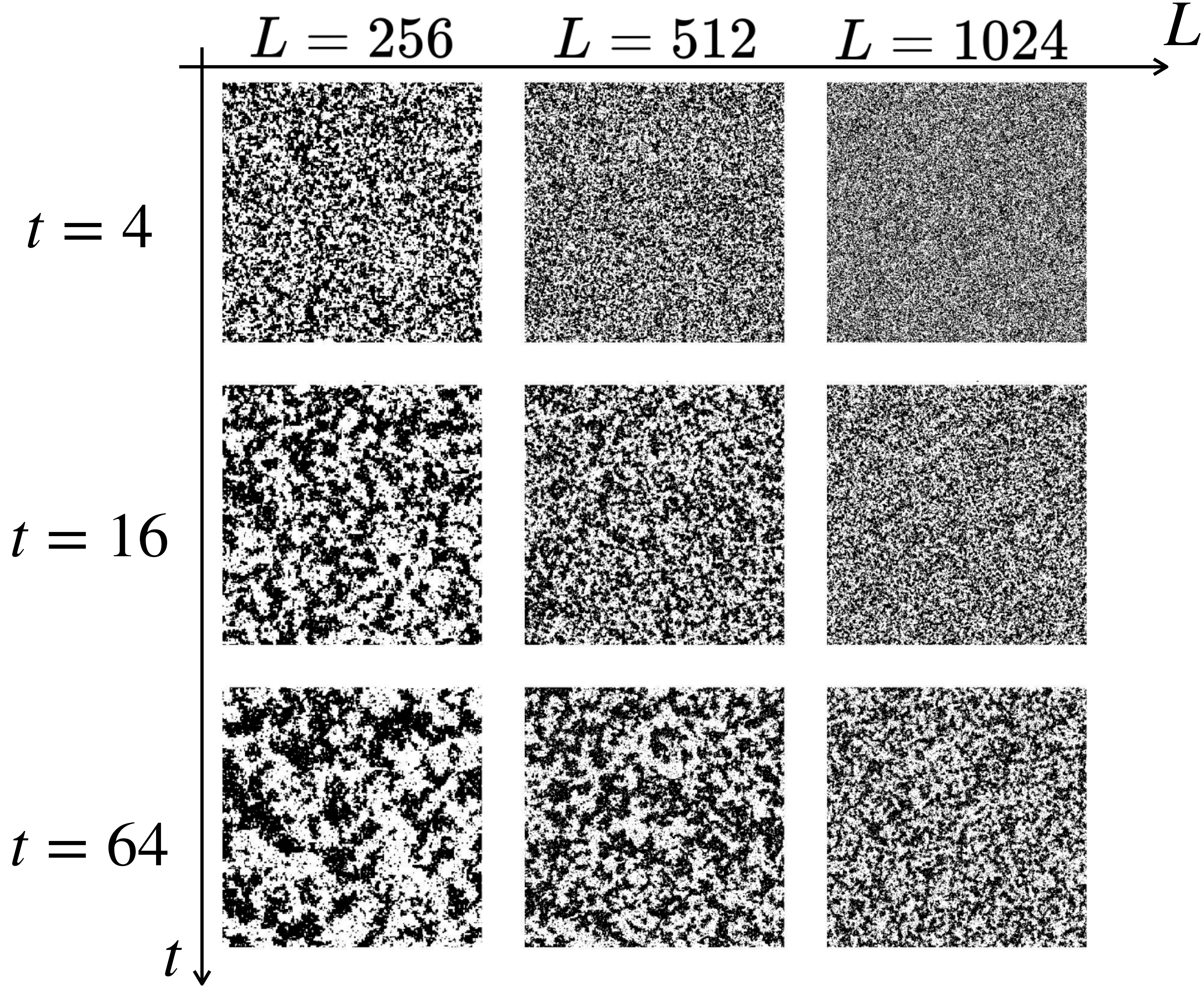
- Detailed balance condition:
  $W_{C,C'} w(C') = W_{C',C} w(C)$



Markov chain Monte Carlo for 2D Ising model

Markov chain Monte Carlo for 2D Ising model at criticality $\beta = (1/2)\log(1+\sqrt{2})$

Initial state: → Random



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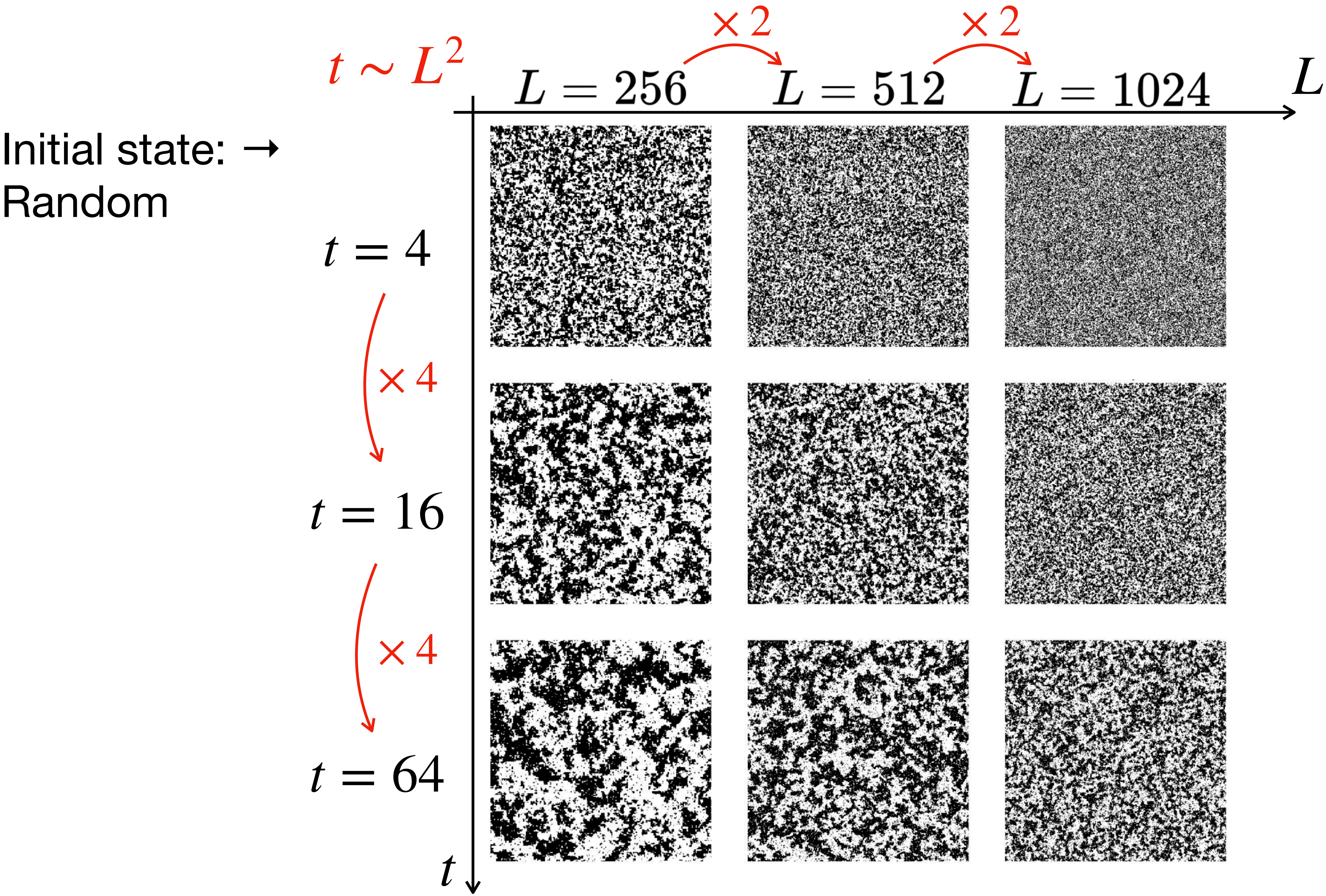

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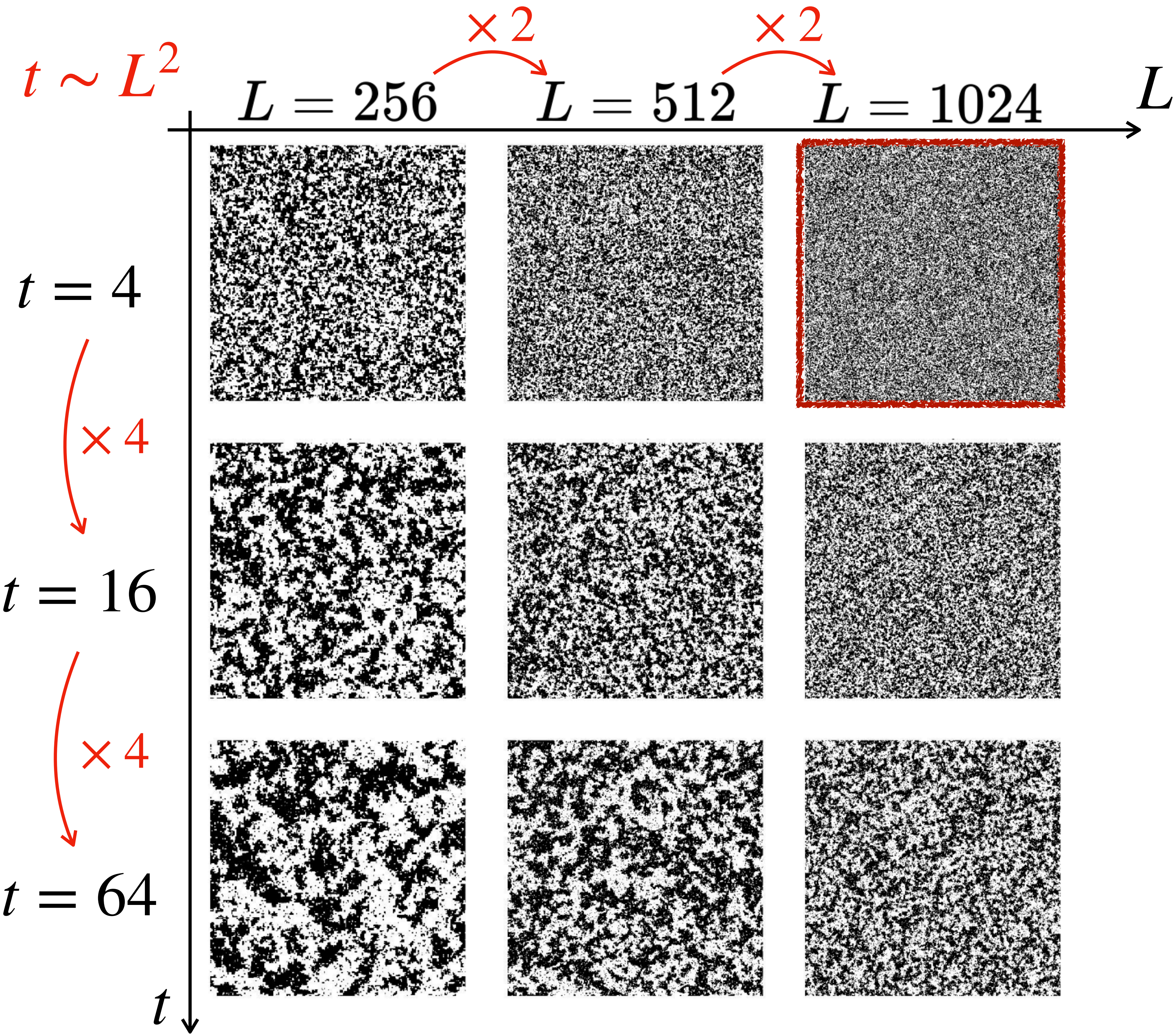

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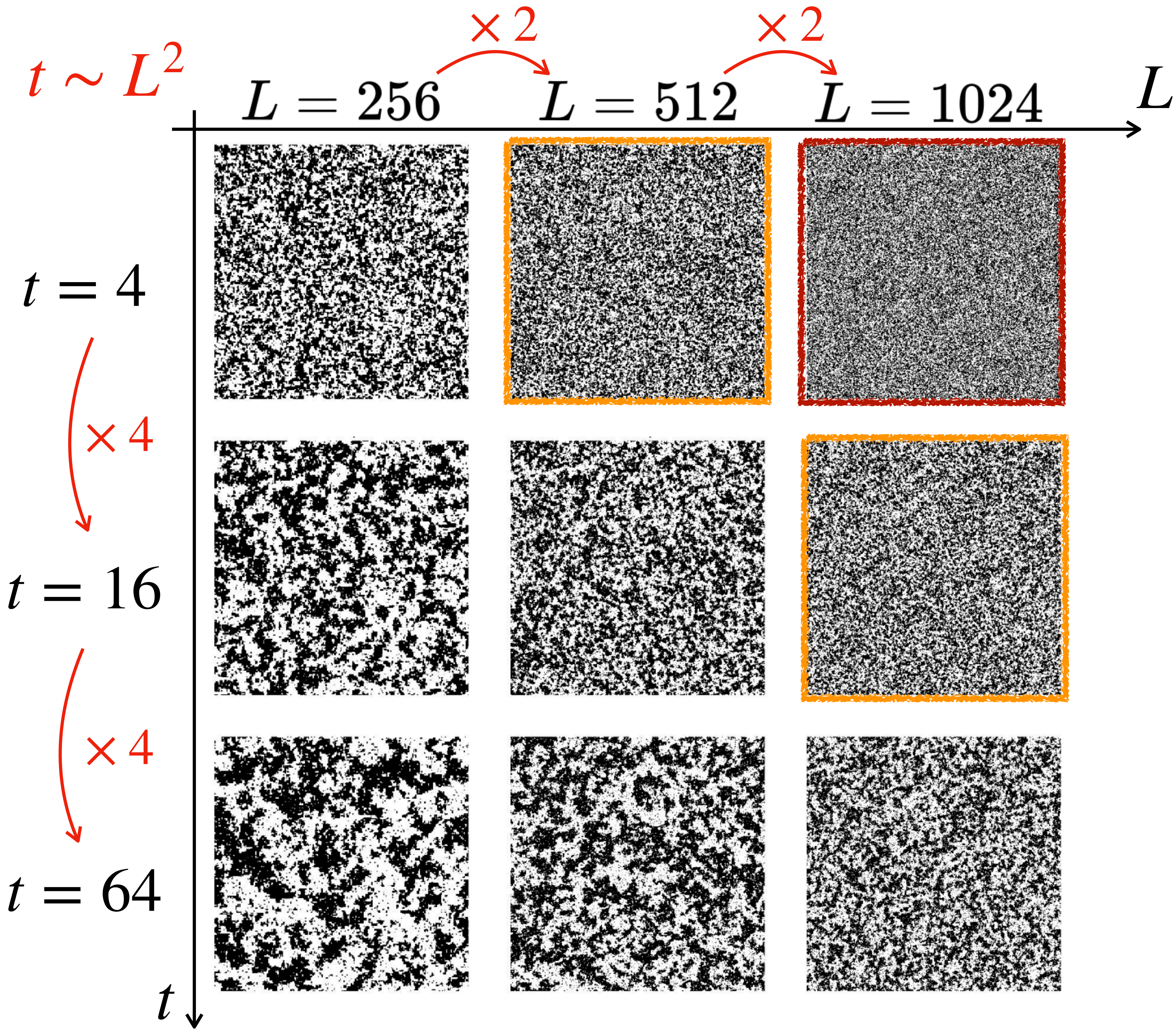

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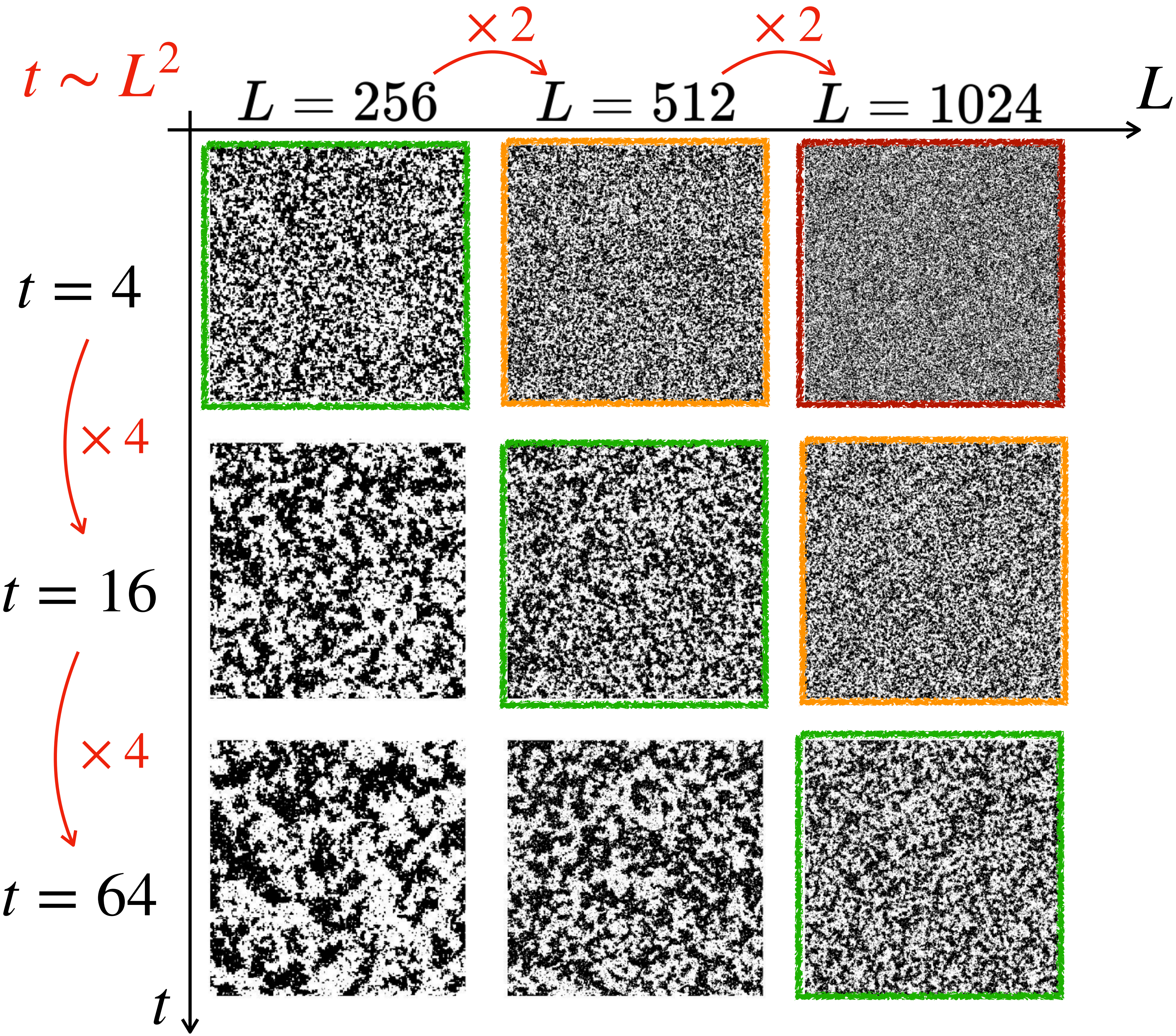

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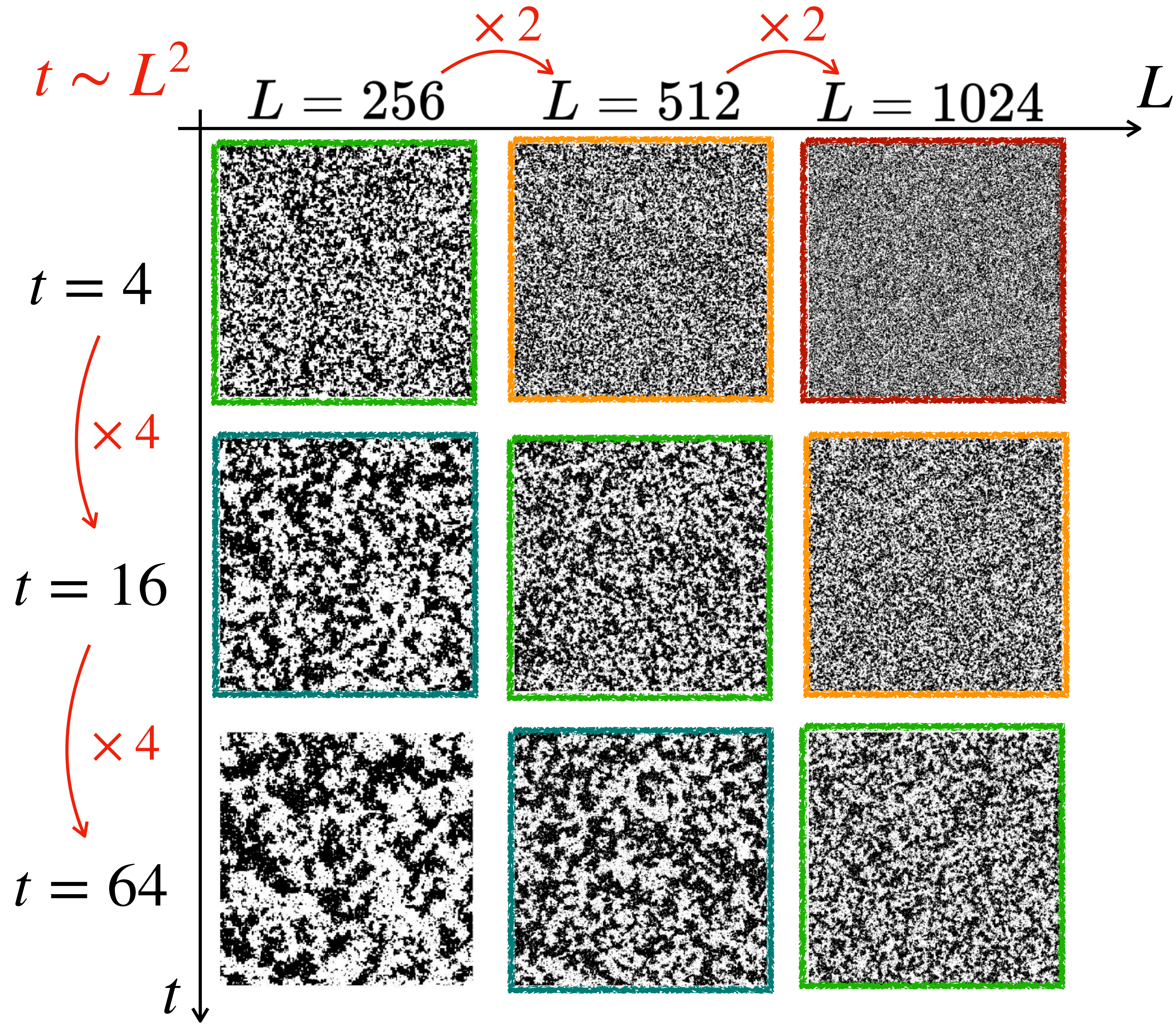

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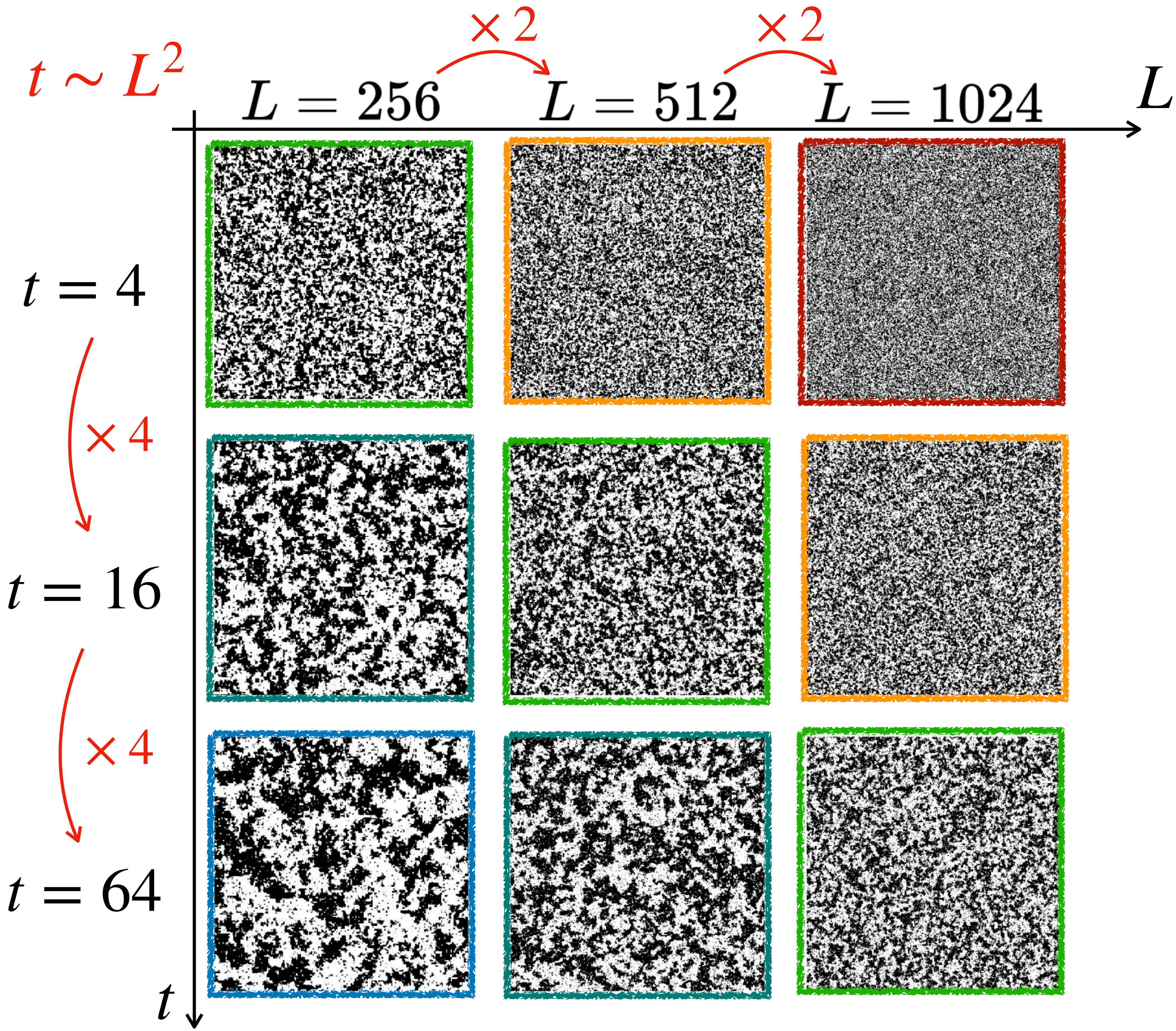
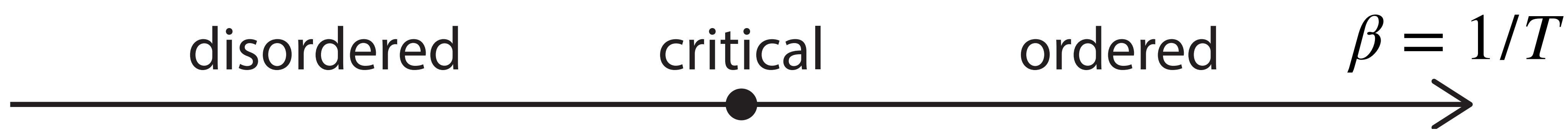


Critical Slowing Down

- As the system approaches to a critical point, the relaxation time τ becomes longer and longer.



- At the critical point, the relaxation time $\tau \propto L^z$ (z : dynamic critical exponent).

$$|\langle O e^{Wt} O\rangle - \langle O\rangle^2| \simeq C e^{-t/\tau} \text{ with } \tau = 1/\epsilon.$$

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disordered critical ordered $\beta = 1/T$

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Models	z
Ising (2D)	2.1667(5) [14]
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Four-state Potts (2D)	2.296(5) [18]
ASEP	3/2 [50, 51]
Wolff algorithm	0.3 [10]

$z \geq 2$ (Ising (2D) through Four-state Potts (2D))

$z < 2$ (ASEP, Wolff algorithm)

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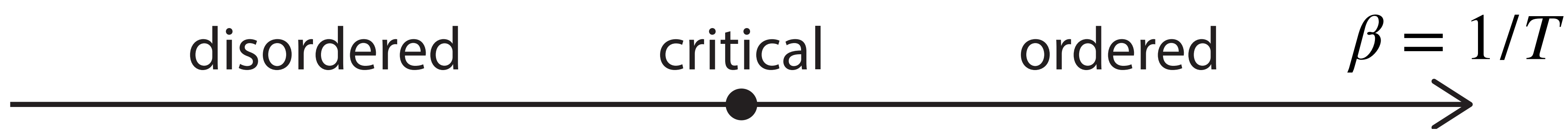

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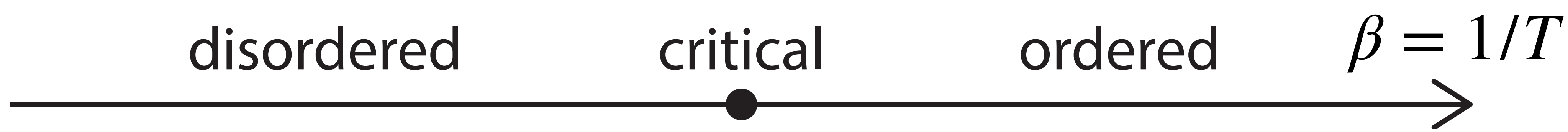

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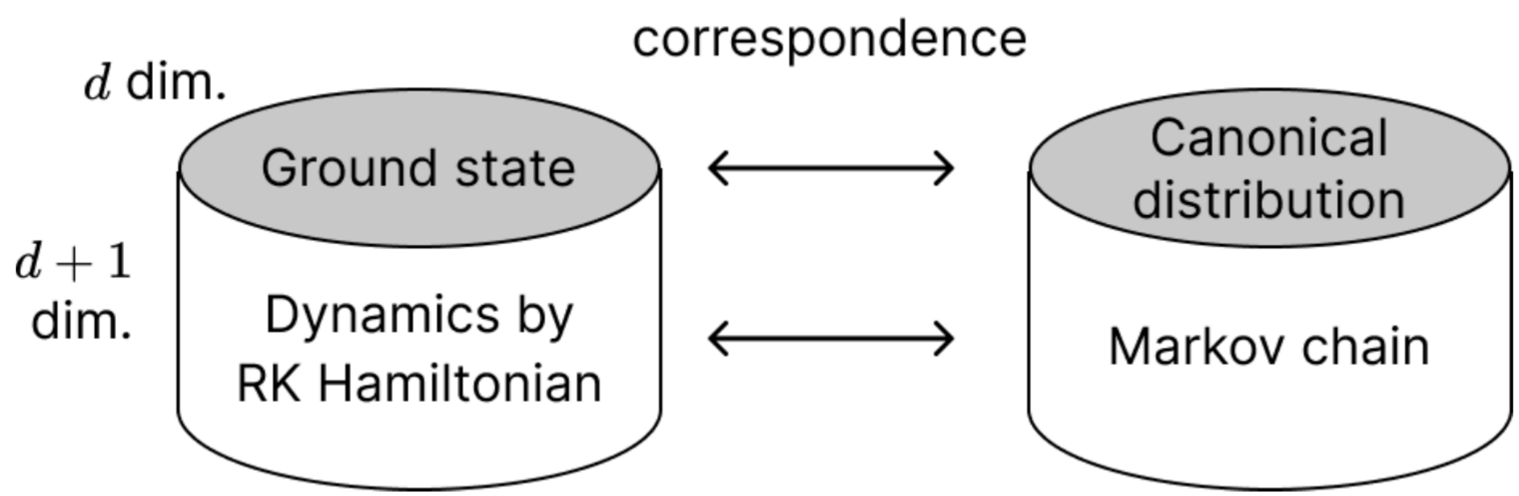
- However, only $z \geq 1.75$ has been proven for 2D Ising model so far.

Mapping to frustration-free Hamiltonian

- Markov chain with detailed balance and locality

can be mapped to FF Hamiltonian by $H_{C,C'} = -\sqrt{\frac{w(C')}{w(C)}} W_{C,C'}$.



$$w(C) = e^{J\sum_{(i,j)} \sigma_i \sigma_j}$$

$$\hat{H}_{\vec{r}} = \frac{1}{2\cosh(J\sum_{\vec{r}'\in B_{\vec{r}'}} \hat{\sigma}^z_{\vec{r}'})}\left(e^{-J\hat{\sigma}^z_{\vec{r}}\sum_{\vec{r}'\in B_{\vec{r}}} \hat{\sigma}^z_{\vec{r}'}} - \hat{\sigma}^x_{\vec{r}}\right)$$ for Ising model

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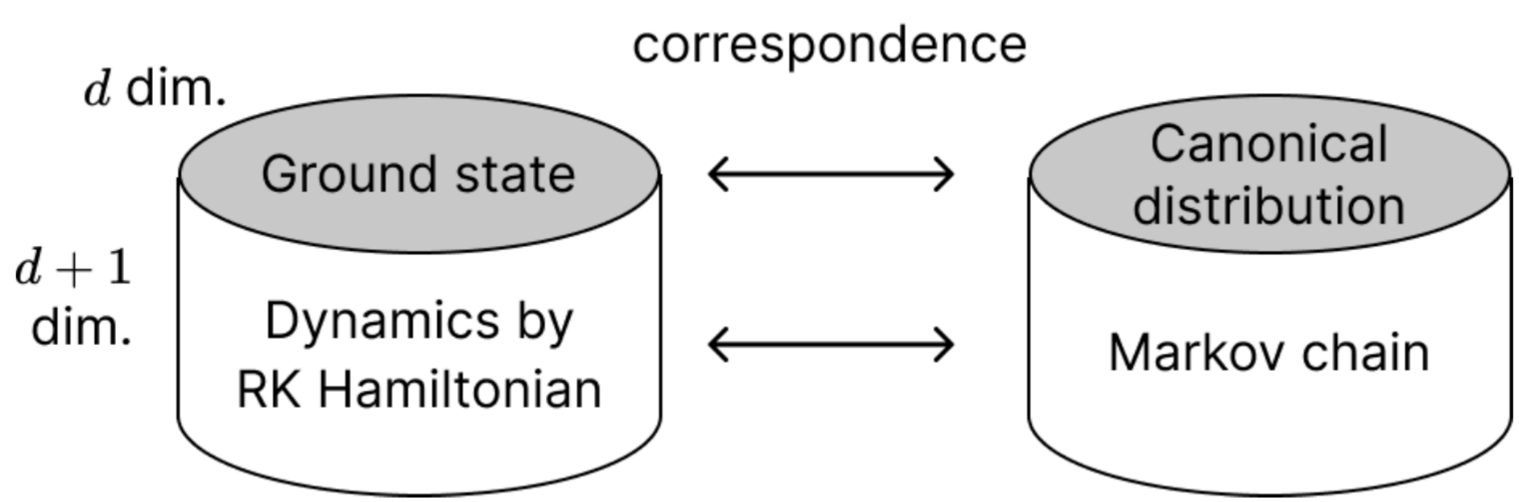

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- Our result on FF Hamiltonian $\epsilon = O((\log L)^2/L^2)$ immediately gives the first ever proof of $z \geq 2$!!

Fermionic Systems

Free Fermions

- When a tight-binding model $\hat{H} = \sum_{i,j \in \Lambda} \hat{c}_i^\dagger t_{ij} \hat{c}_j$ frustration free?

- Do the three conjectures hold?
  - If gapless, finite size gap is $\epsilon \sim L^{-z}$ with $z \geq 2$.
  - If gapped, finite size splitting of degeneracy is absent.
  - No local FF Hamiltonian for Chern insulators.

Tight-binding model

arXiv 2503.12879
arXiv 2503.14312

- $\hat{H} = \sum_{\vec{R}} \hat{H}_{\vec{R}}$ with $\hat{H}_{\vec{R}} = \hat{\vec{c}}^{\dagger}_{\vec{R}} h_{\vec{R}} \hat{\vec{c}}_{\vec{R}} + C_{\vec{R}}$.

$h_{\vec{R}}$ is a Hermitian matrix and $\hat{\vec{c}}_{\vec{R}} = \begin{pmatrix} \hat{c}_{\vec{R}} & \hat{c}_{\vec{R}+\vec{a}} & \cdots \end{pmatrix}$ a finite dimension (= range of hopping) vector.

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- Can be written as $\hat{H}_{\vec{R}} = \underbrace{\sum_{\alpha} \mu_{\vec{R}\alpha} \hat{\psi}^{\dagger}_{\vec{R}\alpha} \hat{\psi}_{\vec{R}\alpha}}_{\hat{H}^{(+)}_{\vec{R}}} + \underbrace{\sum_{\beta} \nu_{\vec{R}\beta} \hat{\phi}_{\vec{R}\beta} \hat{\phi}^{\dagger}_{\vec{R}\beta}}_{\hat{H}^{(-)}_{\vec{R}}}$. → positive semidefinite
  - $h_{\vec{R}} = \sum_{\alpha} \mu_{\vec{R}\alpha} \vec{\psi}_{\vec{R}\alpha} \vec{\psi}^{\dagger}_{\vec{R}\alpha} - \sum_{\beta} \nu_{\vec{R}\beta} \vec{\phi}_{\vec{R}\beta} \vec{\phi}^{\dagger}_{\vec{R}\beta}$
  - $\hat{\psi}_{\vec{R}\alpha} = \vec{\psi}^{\dagger}_{\vec{R}\alpha} \hat{\vec{c}}_{\vec{R}}$ and $\hat{\phi}_{\vec{R}\beta} = \vec{\phi}^{\dagger}_{\vec{R}\beta} \hat{\vec{c}}_{\vec{R}}$ and set $C_{\vec{R}} = \sum_{\beta} \nu_{\vec{R}\beta}$.

Tight-binding model

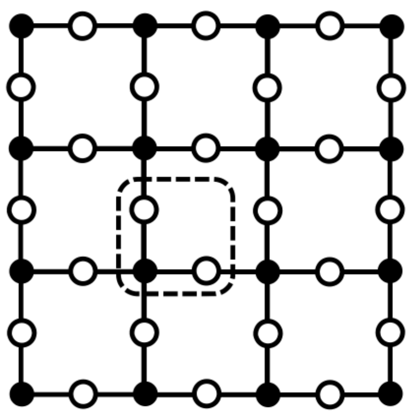
arXiv 2503.12879
arXiv 2503.14312

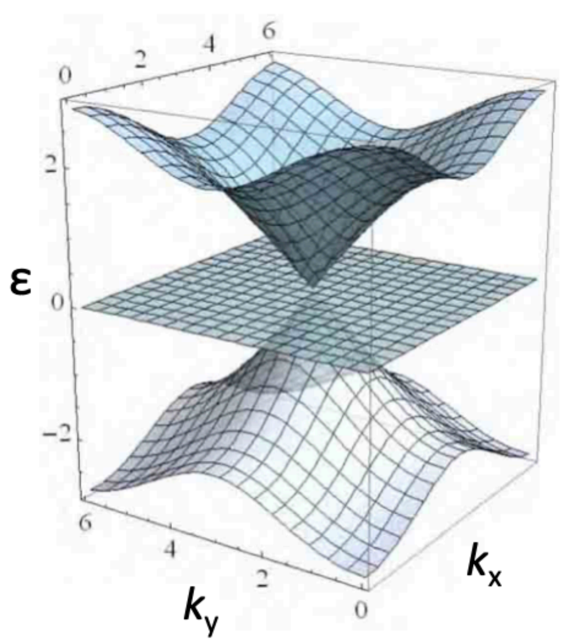
- $\hat{H} = \sum_{\vec{R}} \hat{H}_{\vec{R}}$ with $\hat{H}_{\vec{R}} = \hat{\vec{c}}^{\dagger}_{\vec{R}} h_{\vec{R}} \hat{\vec{c}}_{\vec{R}} + C_{\vec{R}}$.
  $h_{\vec{R}}$ is a Hermitian matrix and $\hat{\vec{c}}_{\vec{R}} = \begin{pmatrix} \hat{c}_{\vec{R}} & \hat{c}_{\vec{R}+\vec{a}} & \cdots \end{pmatrix}$ a finite dimension (= range of hopping) vector.
- Can be written as $\hat{H}_{\vec{R}} = \underbrace{\sum_{\alpha} \mu_{\vec{R}\alpha} \hat{\psi}^{\dagger}_{\vec{R}\alpha} \hat{\psi}_{\vec{R}\alpha}}_{\hat{H}^{(+)}_{\vec{R}}} + \underbrace{\sum_{\beta} \nu_{\vec{R}\beta} \hat{\phi}_{\vec{R}\beta} \hat{\phi}^{\dagger}_{\vec{R}\beta}}_{\hat{H}^{(-)}_{\vec{R}}}$. → positive semidefinite
  - $h_{\vec{R}} = \sum_{\alpha} \mu_{\vec{R}\alpha} \vec{\psi}_{\vec{R}\alpha} \vec{\psi}^{\dagger}_{\vec{R}\alpha} - \sum_{\beta} \nu_{\vec{R}\beta} \vec{\phi}_{\vec{R}\beta} \vec{\phi}^{\dagger}_{\vec{R}\beta}$
  - $\hat{\psi}_{\vec{R}\alpha} = \vec{\psi}^{\dagger}_{\vec{R}\alpha} \hat{\vec{c}}_{\vec{R}}$ and $\hat{\phi}_{\vec{R}\beta} = \vec{\phi}^{\dagger}_{\vec{R}\beta} \hat{\vec{c}}_{\vec{R}}$ and set $C_{\vec{R}} = \sum_{\beta} \nu_{\vec{R}\beta}$.
- Frustration-free if and only if $\hat{\psi}_{\vec{R}\alpha} |\Phi\rangle = \hat{\phi}^{\dagger}_{\vec{R}'\beta} |\Phi\rangle = 0$.
- Equivalently, $\{\hat{\psi}_{\vec{R}\alpha}, \hat{\phi}^{\dagger}_{\vec{R}'\beta}\} = 0$ for all $\vec{R}, \vec{R}', \alpha, \beta$ → $[\hat{H}^{(+)}_{\vec{R}}, \hat{H}^{(-)}_{\vec{R}'}] = 0$.

Examples

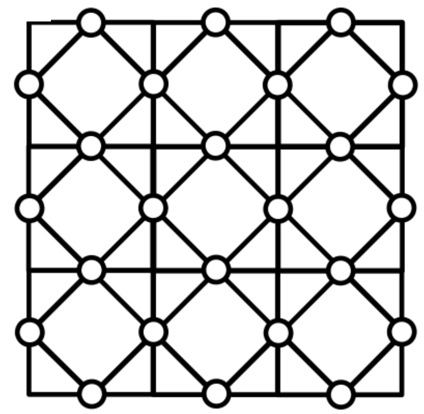
- Flat bands with **quadratic band touching**

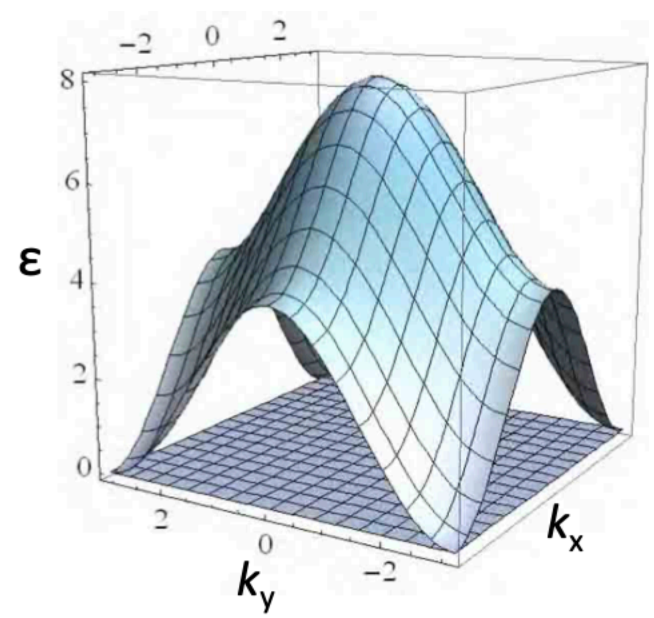
Lieb lattice



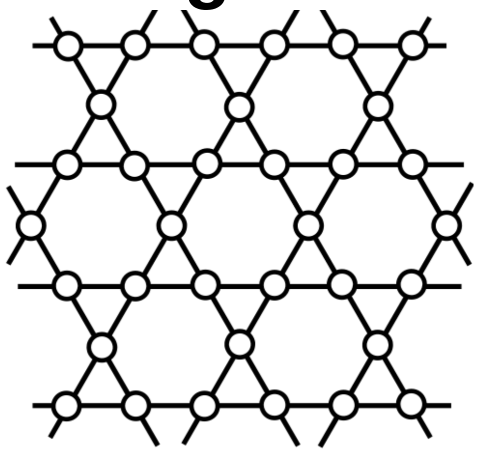


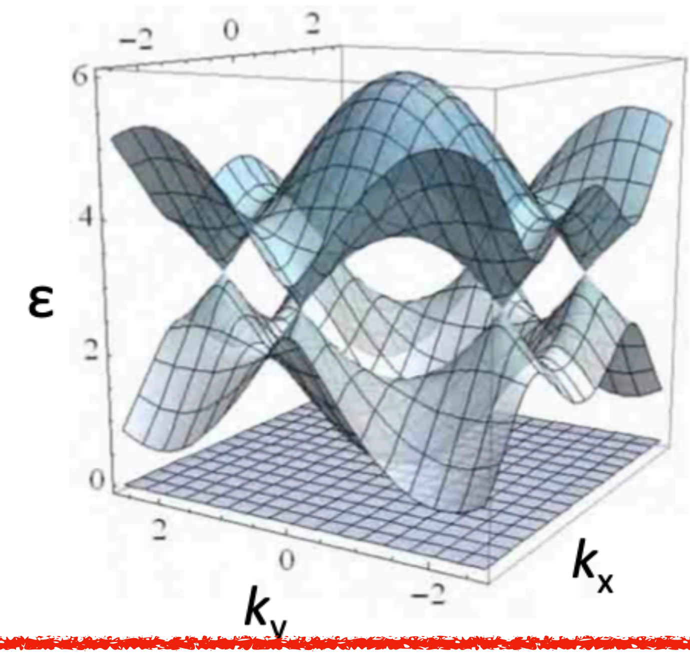
Checkerboard



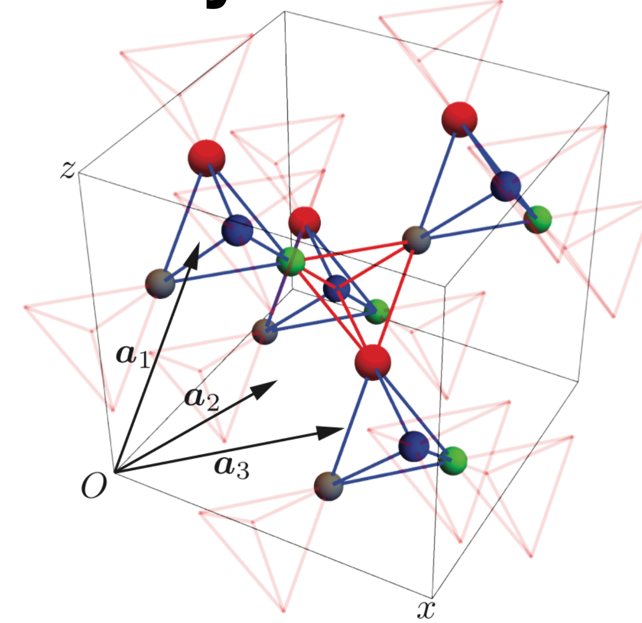


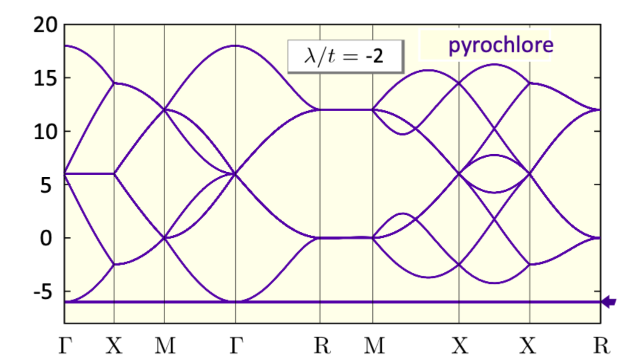
Kagome





Pyrochlore





Nakai-Hotta (2022)

(Images from Katsura&Maruyama 2015)

- Repulsive interaction → Flatband ferromagnets (Mielke, Tasaki, ….)

Frustration-Free Fee Fermions

arXiv 2503.12879
arXiv 2503.14312

- Can be written as $\hat{H}_{\vec{R}} = \underbrace{\sum_{\alpha} \mu_{\vec{R}\alpha} \hat{\psi}^{\dagger}_{\vec{R}\alpha} \hat{\psi}_{\vec{R}\alpha}}_{\hat{H}^{(+)}_{\vec{R}}} + \underbrace{\sum_{\beta} \nu_{\vec{R}\beta} \hat{\phi}_{\vec{R}\beta} \hat{\phi}^{\dagger}_{\vec{R}\beta}}_{\hat{H}^{(-)}_{\vec{R}}}.$

Fourier transform $\hat{H}^{(+)}_{\vec{R}} \downarrow \hat{\vec{c}}^{\dagger}_{\vec{k}} H^{(+)}_{\vec{k}} \hat{\vec{c}}_{\vec{k}}$ $\quad \hat{H}^{(-)}_{\vec{R}} \downarrow \hat{\vec{c}}^{\dagger}_{\vec{k}} H^{(-)}_{\vec{k}} \hat{\vec{c}}_{\vec{k}}$

- FF condition: $\{\hat{\psi}_{\vec{R}\alpha}, \hat{\phi}^{\dagger}_{\vec{R}'\beta}\} = 0$ for all $\vec{R}$, $\vec{R}'$, α, β

→ $[H^{(+)}_{\vec{k}}, H^{(-)}_{\vec{k}}] = 0$

→ $H^{(+)}_{\vec{k}}$ and $H^{(-)}_{\vec{k}}$ can be diagonalized separately. Both produce analytic bands.

→ **Quadratic band touching**. Excitation energy $O(L^{-2})$.

FF decomposition

Kitaev (2006)
for free Majoranas

- Any gapped TB model can be rewritten as FF form
but with exponential tail.

Sengoku-Po-HW (2025)

- $\hat{H} = \sum_{i,j} \hat{c}_i^\dagger h_{i,j} \hat{c}_j = \hat{\vec{c}}^\dagger h \hat{\vec{c}}, \quad h = \sum_{\epsilon_n > 0} \epsilon_n P_n + \sum_{\epsilon_n < 0} \epsilon_n P_n$

- $$\begin{aligned} \hat{H} &= \hat{\vec{c}}^\dagger h^{(+)} \hat{\vec{c}} + \hat{\vec{c}}^\dagger h^{(-)} \hat{\vec{c}} \\ &= \hat{\vec{c}}^\dagger h^{(+)} \hat{\vec{c}} + \hat{\vec{c}}^T (-h^{(-)T}) \left(\hat{\vec{c}}^\dagger\right)^T + E_0 \\ &= \hat{\vec{c}}^\dagger \sqrt{h^{(+)}} \sqrt{h^{(+)}} \hat{\vec{c}} + \hat{\vec{c}}^T \sqrt{-h^{(-)T}} \sqrt{-h^{(-)T}} (\hat{\vec{c}}^\dagger)^T + E_0 \\ &= \sum_i \left(\underline{(\hat{\psi}_i^{(+)})^\dagger \hat{\psi}_i^{(+)} + \hat{\psi}_i^{(-)} (\hat{\psi}_i^{(-)})^\dagger} \right) \text{ with } \hat{\psi}_i^{(\pm)} = \sum_j \left(\sqrt{\pm h^{(\pm)}} \right)_{i,j} \hat{c}_j \end{aligned}$$

→ annihilates ground state!

FF decomposition

Kitaev (2006)
for free Majoranas

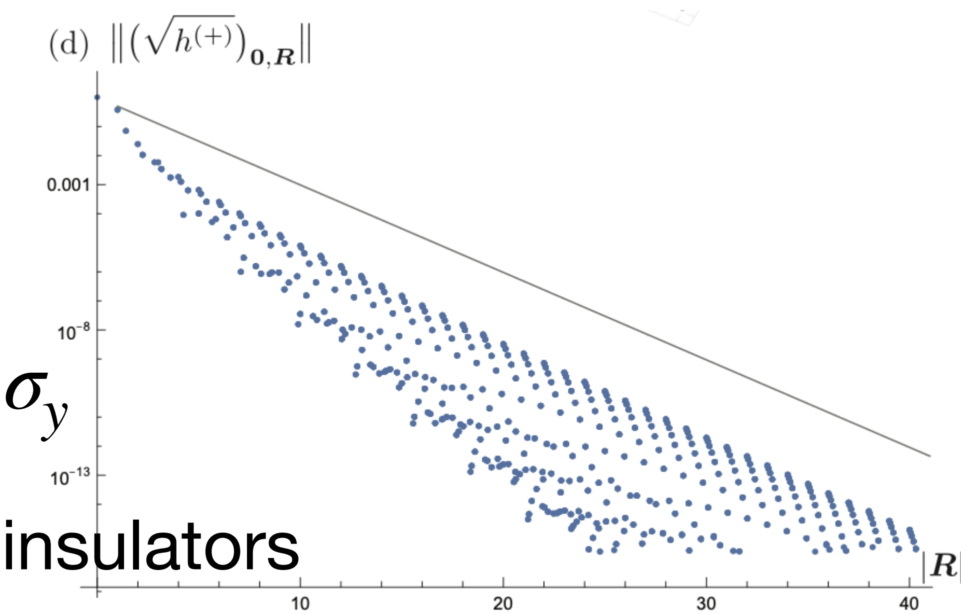
Sengoku-Po-HW (2025)

- Any gapped TB model can be rewritten as FF form
  <u>but with exponential tail.</u>

- $\hat{H} = \sum_{i,j} \hat{c}_i^\dagger h_{i,j} \hat{c}_j = \hat{\vec{c}}^\dagger h \hat{\vec{c}}, \quad h = \sum_{\epsilon_n > 0} \epsilon_n P_n + \sum_{\epsilon_n < 0} \epsilon_n P_n$

- $$\begin{aligned} \hat{H} &= \hat{\vec{c}}^\dagger h^{(+)} \hat{\vec{c}} + \hat{\vec{c}}^\dagger h^{(-)} \hat{\vec{c}} \\ &= \hat{\vec{c}}^\dagger h^{(+)} \hat{\vec{c}} + \hat{\vec{c}}^T (-h^{(-)T}) \left(\hat{\vec{c}}^\dagger\right)^T + E_0 \\ &= \hat{\vec{c}}^\dagger \sqrt{h^{(+)}} \sqrt{h^{(+)}} \hat{\vec{c}} + \hat{\vec{c}}^T \sqrt{-h^{(-)T}} \sqrt{-h^{(-)T}} (\hat{\vec{c}}^\dagger)^T + E_0 \\ &= \sum_i \Big(\underbrace{(\hat{\psi}_i^{(+)})^\dagger \hat{\psi}_i^{(+)} + \hat{\psi}_i^{(-)} (\hat{\psi}_i^{(-)})^\dagger}_{\rightarrow \text{ annihilates ground state!}} \Big) \text{ with } \hat{\psi}_i^{(\pm)} = \sum_j \left(\sqrt{\pm h^{(\pm)}} \right)_{i,j} \hat{c}_j \end{aligned}$$

- e.g. $\hat{H} = \sum_{\vec{k}} \begin{pmatrix} \hat{c}_{\vec{k}1}^\dagger & \hat{c}_{\vec{k}2}^\dagger \end{pmatrix} h_{\vec{k}} \begin{pmatrix} \hat{c}_{\vec{k}1} \\ \hat{c}_{\vec{k}2} \end{pmatrix}$
  with $h_{\vec{k}} = \sin k_x \sigma_x + \sin k_y \sigma_y + (m - \cos k_x - \cos k_y) \sigma_y$



- Exponentially decaying tail in hopping even for Chern insulators

Summary

- Explored general properties of frustration-free systems.

- Three recent conjectures on FF systems
  - If FF & gapless, finite size gap is $\epsilon \sim L^{-z}$ with $z \geq 2$.
    → Dynamic critical exponent in 2D Ising model at criticality.
  - If FF & gapped, finite size splitting of degeneracy is absent.
  - No local FF Hamiltonian for Chern insulators.

- Counter examples for the cojectures