# Characterising Multimode Quantum Correlations in Cavity-Based Quantum Optics

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Quantum properties of light in Cavity-based Quantum Optical Systems :
Microrings, Optomechanics, ...

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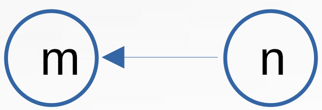
# Quadratic Ham.

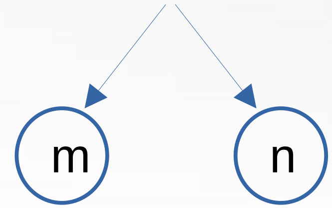
$$\hat{H} = \hbar \sum_{m,n} \boxed{G_{m,n}} \hat{a}^\dagger_m \hat{a}_n + \frac{\hbar}{2} \sum_{m,n} \left[ \boxed{F_{m,n}} \hat{a}^\dagger_m \hat{a}^\dagger_n + \mathrm{h.c.} \right]$$

Mode hopping




- Detuning
- Linear dispersion
- FC processes
- SPM and XPM

Pair generation




- SPDC in 2nd and 3rd Order interactions

# Quadratic Ham.

PhLAM
Physique des Lasers Atomes et Molécules

$$\hat{H} = \hbar \sum_{m,n} \boxed{G_{m,n}} \hat{a}_m^\dagger \hat{a}_n + \frac{\hbar}{2} \sum_{m,n} \left[ \boxed{F_{m,n}} \hat{a}_m^\dagger \hat{a}_n^\dagger + \text{h.c.} \right]$$

Mode hopping

Pair generation

Optomechanical System

$$\mathrm{G} = \begin{pmatrix} \Delta & g \\ g & \omega_m \end{pmatrix}, \quad \mathrm{F} = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$$

Dual Pump SFWM MRR

$$G = -\begin{pmatrix} |\beta_{p1}|^2 + |\beta_{p2}|^2 - \Delta & \beta_{p1}\beta_{p2}^* & 0 \\ \beta_{p1}^*\beta_{p2} & |\beta_{p1}|^2 + |\beta_{p2}|^2 - \Delta & \beta_{p1}\beta_{p2}^* \\ 0 & \beta_{p1}^*\beta_{p2} & |\beta_{p1}|^2 + |\beta_{p2}|^2 - \Delta \end{pmatrix}$$

$$F = -\begin{pmatrix} 0 & \beta_{p1}^2 & 2\beta_{p1}\beta_{p2} \\ \beta_{p1}^2 & \beta_{p1}\beta_{p2} & \beta_{p2}^2 \\ 2\beta_{p1}\beta_{p2} & \beta_{p2}^2 & 0 \end{pmatrix}$$

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# Bogo-de Gennes

Heisenberg picture

collect $\boldsymbol{\xi} = \left(a_1, \ldots, a_N | a_1^\dagger, \ldots, a_N^\dagger\right)^{\mathrm{T}}$

$$i\hbar \frac{d}{dt}\hat{\xi} = \left[\hat{\xi}, \hat{H}\right] \quad \Rightarrow \quad \frac{d}{dt}\hat{\xi} = -i\boxed{K\mathcal{H}}\,\hat{\xi}$$

Diagonalize?

# Bogo-de Gennes

Heisenberg picture

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Bogo-Valatin transf. $B\Lambda B^{-1}$

$\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_N | -\lambda_1, \ldots, -\lambda_N\}$ Real eigenvalues

$$B = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \beta^* & \alpha^* \end{array}\right)$$

Symplectic

# Bogo-de Gennes

Heisenberg picture

collect $\boldsymbol{\xi} = (a_1, \ldots, a_N | a_1^\dagger, \ldots, a_N^\dagger)^{\mathrm{T}}$

$$i\hbar \frac{d}{dt}\hat{\xi} = \left[\hat{\xi}, \hat{H}\right] \quad \Rightarrow \quad \frac{d}{dt}\hat{\xi} = -i\boxed{K\mathcal{H}}\hat{\xi}$$

Bogo-Valatin transf. $B \Lambda B^{-1}$

Bogoliubov modes:

$$\boldsymbol{\zeta} = B^{-1}\boldsymbol{\xi} = (b_1, \ldots, b_N | b_1^\dagger, \ldots, b_N^\dagger)^{\mathrm{T}}$$

$$H = \sum_m \hbar\lambda_m b_m^\dagger b_m + \text{const}$$

B is symplectic not unitary

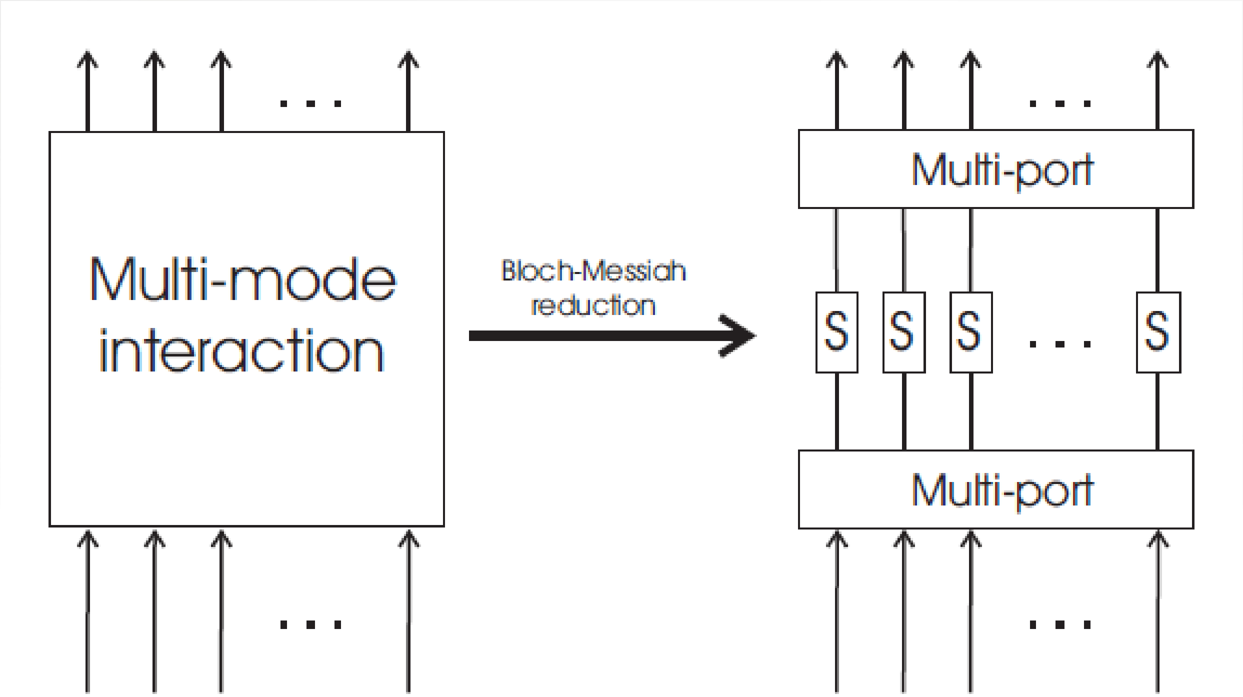
# Bloch-Messiah

Write the formal solution (time-indep. Hamiltonian)

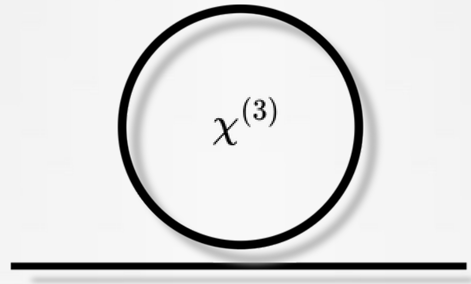
$$\hat{\xi}(t) = \boxed{e^{-iK\mathcal{H}\,t}}\hat{\xi}(0)$$

$$U D V^{\dagger}$$

Always exists



PhLAM Physique des Lasers Atomes et Molécules




# LQLE

$$\underline{\hat{\mathrm{R}}} = \begin{pmatrix} \vdots \\ \hat{x}_{-1} \\ \hat{x}_0 \\ \hat{x}_{+1} \\ \cdots \\ \hline \cdots \\ \hat{y}_{-1} \\ \hat{y}_0 \\ \hat{y}_{+1} \\ \cdots \end{pmatrix}$$

$$\frac{d}{dt}\underline{\hat{\mathrm{R}}}(t) = \left(-\boxed{\Gamma} + \boxed{\mathcal{M}}\right)\underline{\hat{\mathrm{R}}}(t) + \sqrt{2\Gamma}\,\boxed{\underline{\hat{\mathrm{R}}}_{\mathrm{in}}(t)}$$

dissipation

input fluct.
(vacuum)

Effective quadratic Hamiltonian

$$\mathcal{M} = \left(\begin{array}{c|c} \mathrm{Im}\,[G+F] & \mathrm{Re}\,[G-F] \\ \hline -\mathrm{Re}\,[G+F] & -\mathrm{Im}\,[G+F]^{\mathrm{T}} \end{array}\right)$$

$$\hat{H} = \hbar\sum_{m,n} G_{m,n}\hat{a}_m^\dagger\hat{a}_n + \frac{\hbar}{2}\sum_{m,n}\left[F_{m,n}\hat{a}_m^\dagger\hat{a}_n^\dagger + \mathrm{h.c.}\right]$$

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# Formal solution of LQLE

In Fourier domain and with the input-output theory:

$$\boxed{\underline{\hat{\mathrm{R}}}_{\mathrm{out}}(\omega)} = \boxed{S(\omega)}\boxed{\underline{\hat{\mathrm{R}}}_{\mathrm{in}}(\omega)}$$

FT of *bona fide* quadrature operators

FT of *bona fide* quadrature operators

Transfer function

$$S(\omega) = \sqrt{2\Gamma}\left(\mathrm{i}\omega\mathbb{I} + \Gamma - \mathcal{M}\right)^{-1}\sqrt{2\Gamma} - \mathbb{I}$$

Must be ω-symplectic!

# Decoupling the output

Find linear combinations of modes such that:

1. are “normal”
2. define physical observables
3. treat smoothly time and frequency

---

Analytic Bloch-Messiah decomposition (ABMD)
[Gouzien, PRL **125**, 103601 (2020)]:

$$\forall\, \omega \in \mathbb{R} : \qquad \exists\, S(\omega) = U(\omega) D(\omega) V^{\dagger}(\omega)$$

with

| | |
|---|---|
| $U(\omega),\ V(\omega)$ | Smooth unitary and omega-symplectic |
| $D(\omega)$ | Smooth diagonal (squeezing and anti-squeezing) |

# PhLAM Physique des Lasers Atomes et Molécules Decoupling the output

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**ATTENTION!** ABMD is different from a point-by-point BMD (that does not satisfy pt. 3)

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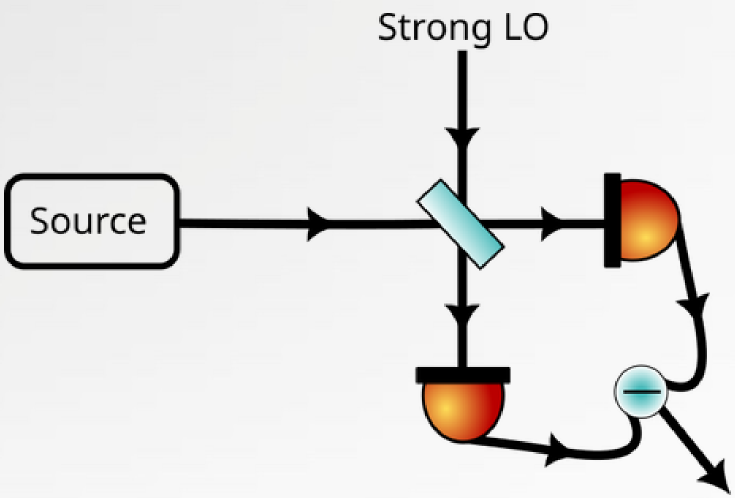
# Homodyne Detection




$$E_{\mathrm{LO}}(t) = i\sum_m \boxed{\alpha_m} e^{-i\omega_{s,m}t} + \mathrm{H.c.},$$

$$\vec{Q} = \Big(\mathrm{Re}(\alpha_1), \ldots, \mathrm{Re}(\alpha_N) | \mathrm{Im}(\alpha_1), \ldots, \mathrm{Im}(\alpha_N)\Big)^{\mathrm{T}}$$

Photocurrent

$$\hat{\imath}(t) \propto Q^{\mathrm{T}} \hat{\boldsymbol{R}}_{\mathrm{out}}(t)$$

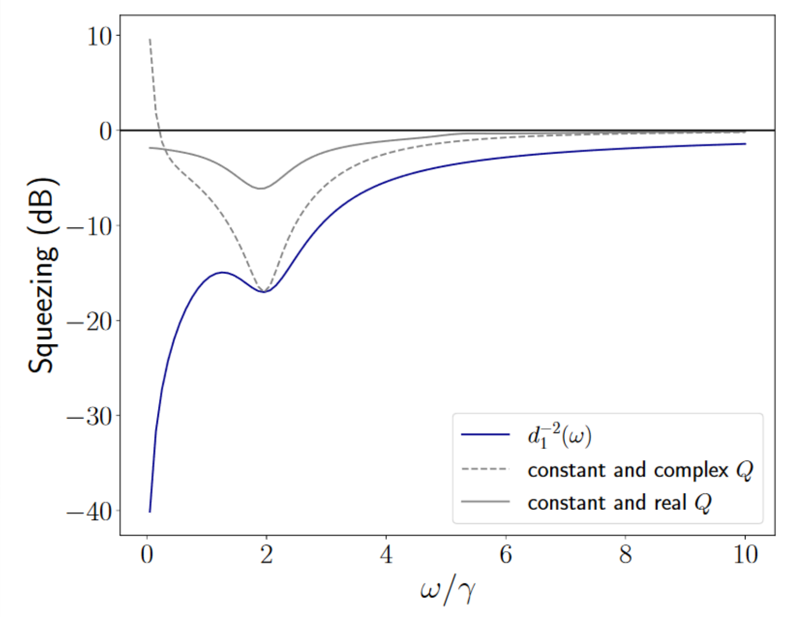
Noise spectral power

$$\Sigma_Q(\omega) = \boldsymbol{Q}^T \sigma_{\mathrm{out}}(\omega) \boldsymbol{Q}$$

# Mode-matching

$$\Sigma_{\mathbf{Q}}(\omega) = \frac{1}{2\sqrt{2\pi}} \left[\mathbf{Q}^{\mathrm{T}} \underline{U(\omega)\right] D^2(\omega) \left[U^\dagger(\omega)}\mathbf{Q}\right]$$

$$\sigma_{\mathrm{out}}(\omega)$$




Gouzien, PRR **5**, 025178 (2023)

# Mode-matching

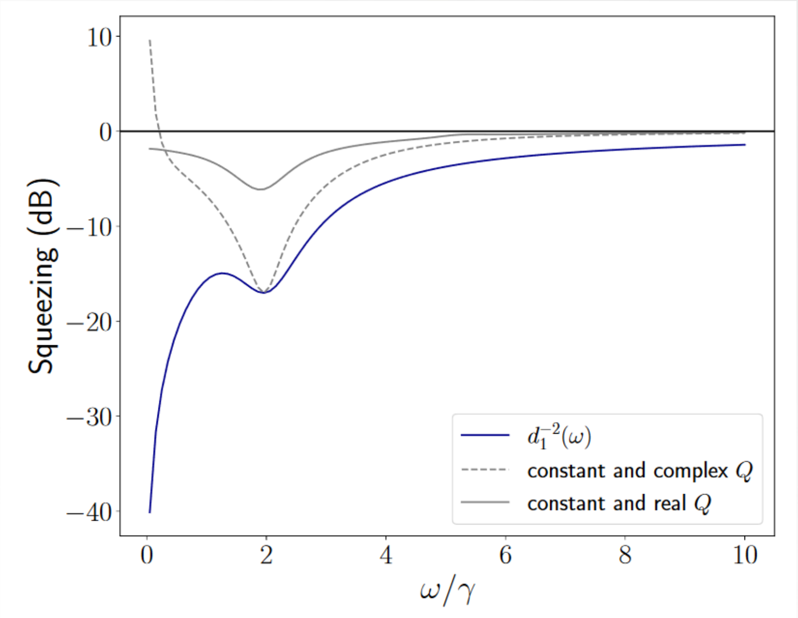
$$\Sigma_Q(\omega) = \frac{1}{2\sqrt{2\pi}} \left[\boxed{Q^T U(\omega)}\right] D^2(\omega) \left[U^\dagger(\omega) Q\right]$$

Mode-matching:

HD Local Oscillator Q
- Constant
- Real

Supermodes $U(\omega)$
- Morphing
- Complex

Hidden/complex squeezing

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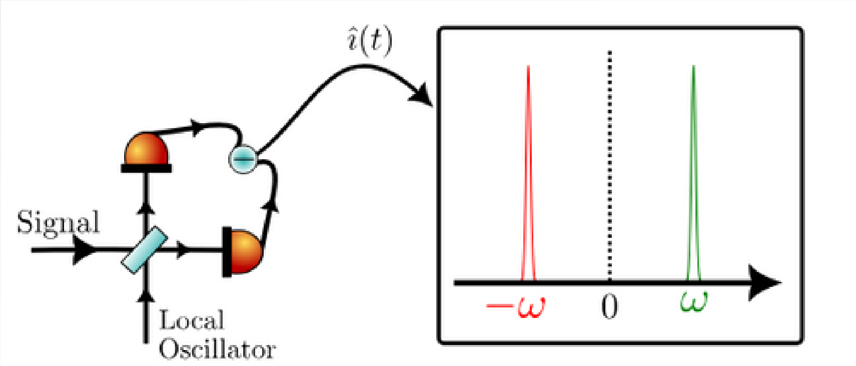
# Hidden squeezing

Hidden or complex squeezing

1. Resonator detection [Barbosa2013]
2. Synodyne detection [Buchmann2016]

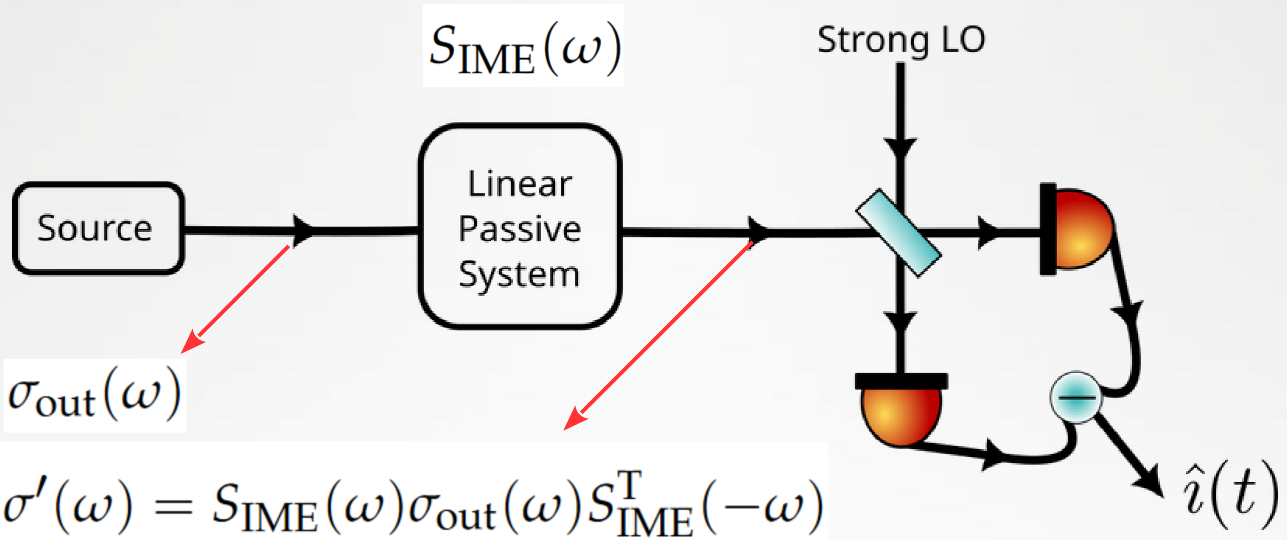



Standard squeezing $\sigma(\omega) = \sigma(-\omega)$

Hidden/Complex squeezing $\sigma(\omega) \neq \sigma(-\omega)$

Dioum, Phys. Rev. A **112**, 023705 (2025)

# Interf. with memory effect




$$\sigma'(\omega) = S_{\mathrm{IME}}(\omega)\sigma_{\mathrm{out}}(\omega)S_{\mathrm{IME}}^{\mathrm{T}}(-\omega)$$

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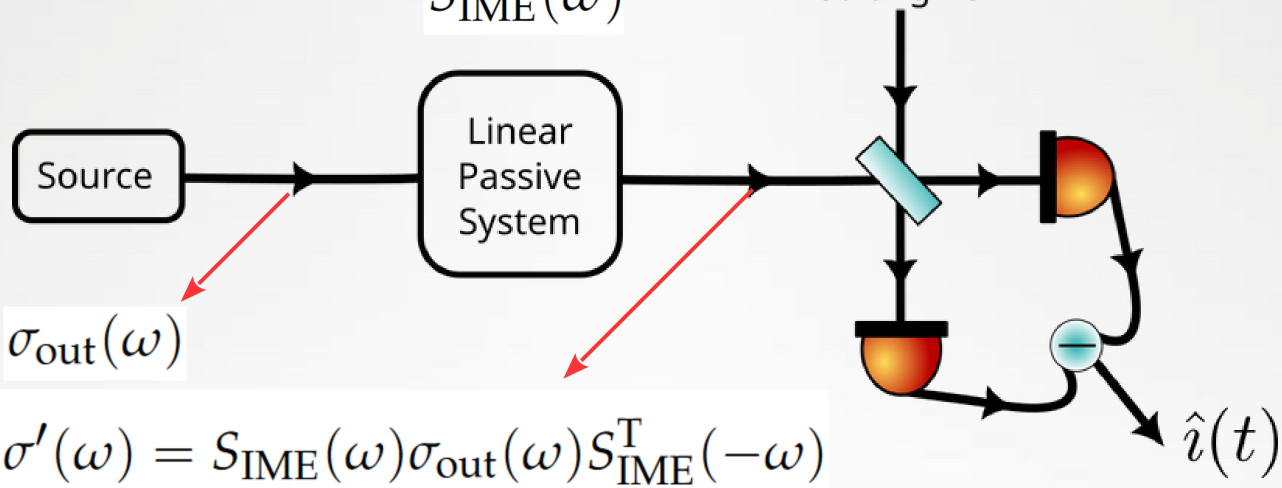
# Interf. with memory effect

$S_{\mathrm{IME}}(\omega)$




$\sigma_{\mathrm{out}}(\omega)$

$$\sigma'(\omega) = S_{\mathrm{IME}}(\omega)\sigma_{\mathrm{out}}(\omega)S_{\mathrm{IME}}^{\mathrm{T}}(-\omega)$$

$$\Sigma_Q(\omega) = \frac{1}{2\sqrt{2\pi}}\left[\boxed{\tilde{Q}^{\dagger}(\omega)}U(\omega)\right] D^2(\omega)\left[U^{\dagger}(\omega)\tilde{Q}(\omega)\right]$$

Generalized LO $\quad \tilde{Q}^{\dagger}(\omega) = Q^{\mathrm{T}} S_{\mathrm{IME}}(\omega)$

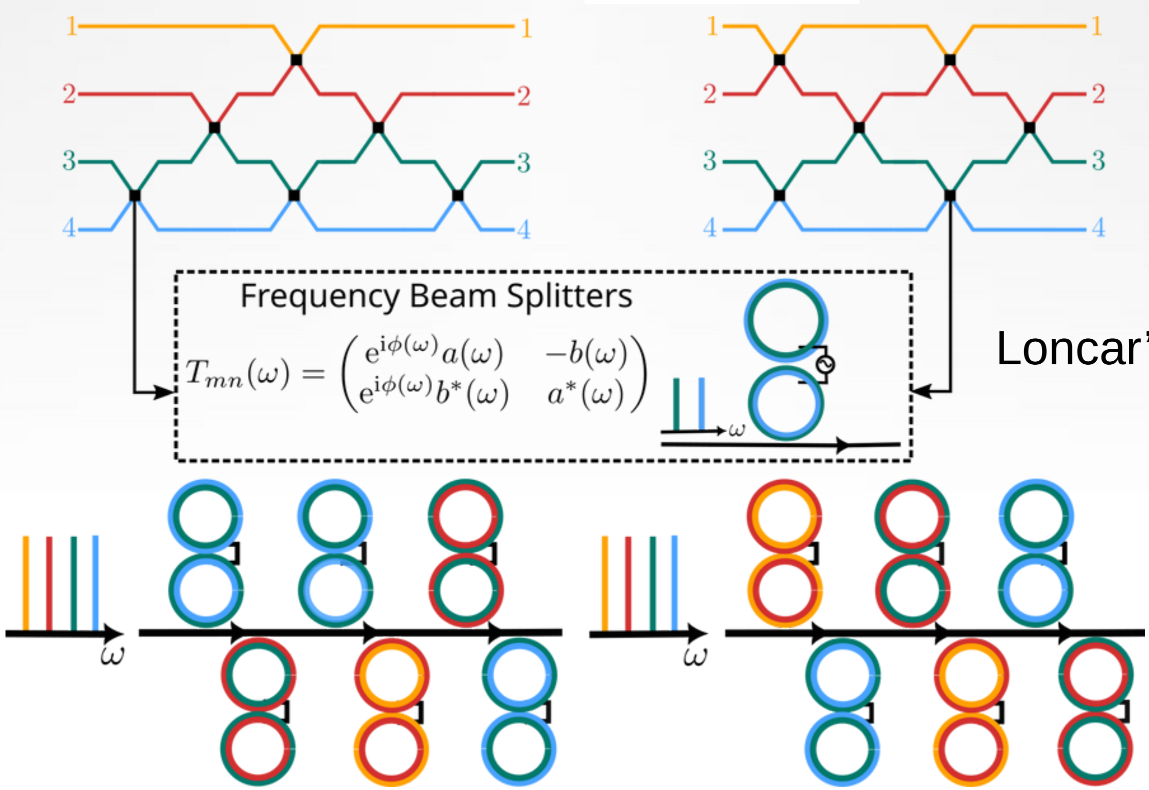
Dioum, Optica Quantum (2024)

# Smooth two-mode decomp.

Two-mode frequency-dependent unitaries

$$S_{\mathrm{IME}}(\omega) \longrightarrow \prod_{(m,n)} T_{mn}(\omega)$$

N(N-1)/2




Loncar's group

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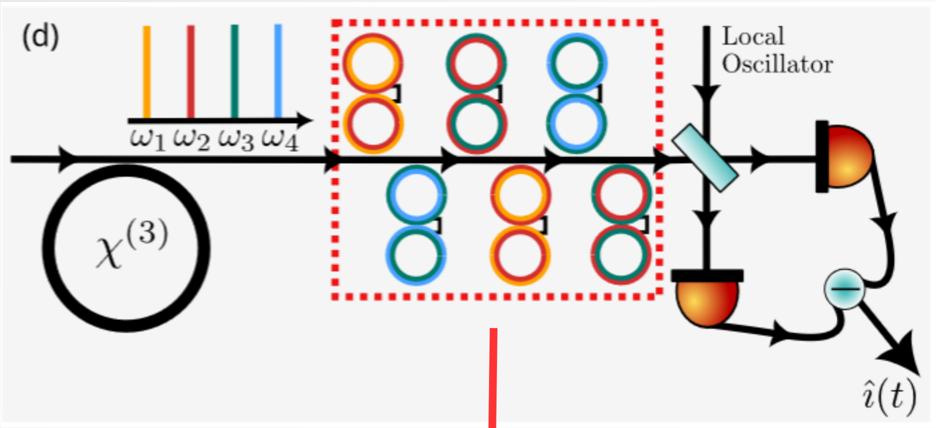
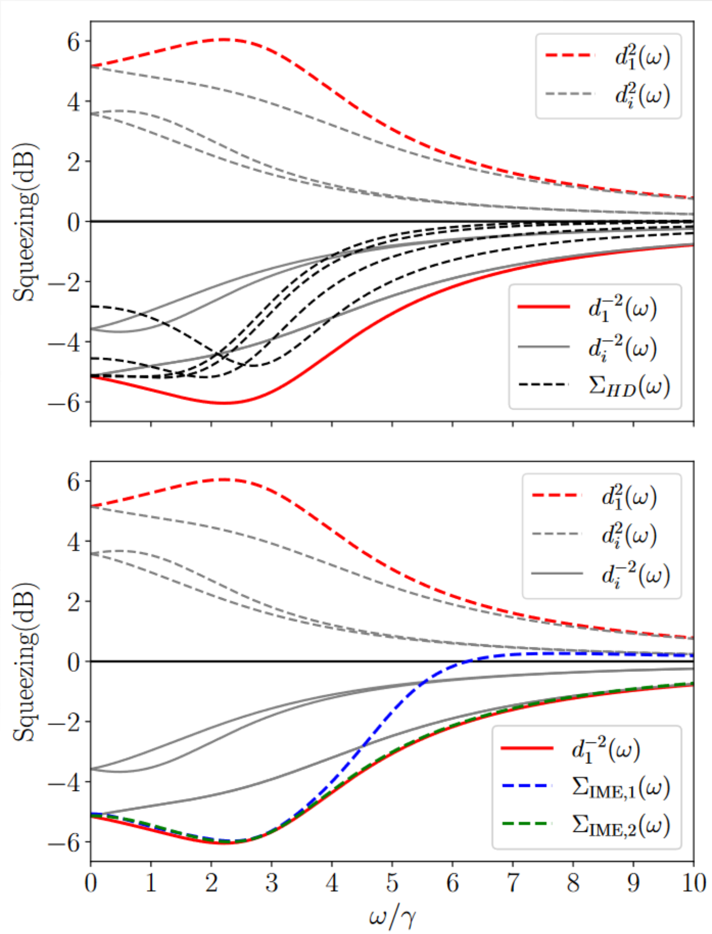
# Some results




Interferometer
with memory effect
(IME)




Dioum, Optica Quantum (2024)

# Conclusions&Perspectives

- General method for complete characterization of quant. fluct. in time&freq.
  Gouzien, PRL **125**, 103601 (2020)

- Complex morphing supermodes → mode-matching problem for HD
  Gouzien, PRR **5**, 025178 (2023)

  1. Criteria for predicting or engineering hidden correlations
     Dioum, PRA **112**, 023705 (2025)

  2. Interferometers with memory effect with new smooth decompositions of frequency dependent unitaries
     Dioum, Optica Quantum (2024)

- Quantum prop. of the light generated in active devices (QCL and ICL)

# Annexes

# Symplectic diag ?

Does there exist a unitary and symplectic diagonalization?

$$i\hbar \frac{d}{dt}\hat{\xi} = \left[\hat{\xi}, \hat{H}\right] \quad \Rightarrow \quad \frac{d}{dt}\hat{\xi} = -i\boxed{K\mathcal{H}}\hat{\xi}$$

$$U\Lambda U^\dagger$$

Yes, but only if either $G = 0$ or $F = 0$

# Driven-dissipative

More than quadratic Hamiltonian $\longrightarrow$ Nonlinear Quantum Langevin eqs. $\longrightarrow$ Linear Quantum Langevin eqs.

Nonlinear Quantum Langevin eqs. $\xrightarrow{\text{classical}}$ Coupled mode theory / Total field

Coupled mode theory / Total field $\xrightarrow{\hat{a} \approx \langle \hat{a} \rangle + \widehat{\delta a}}$ Linear Quantum Langevin eqs.

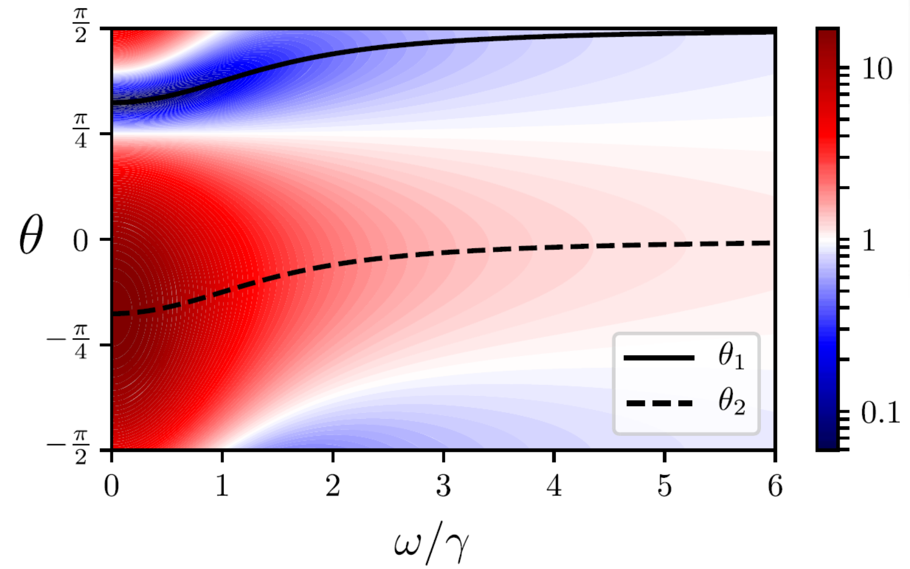
$$\hat{a} \approx \langle \hat{a} \rangle + \widehat{\delta a}$$

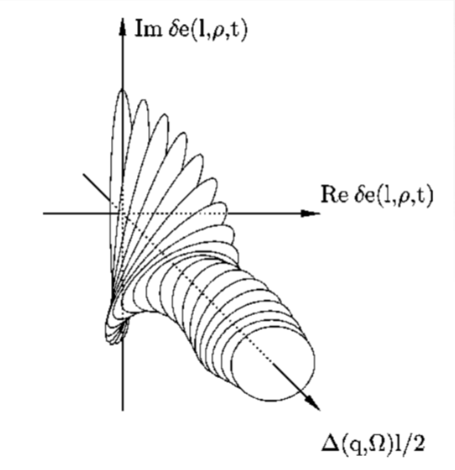
- Coupled mode theory
- Total field: Lugiato-Lefever eqs, driven dissipative Gross-Pitaevskii

# Morphing supermodes

Frequency-dependent squeezing within single-mode broadband beams

1. single mode OPO [Fabre1989,Fabre1990]
2. ponderomotive squeezing [Fabre1994, Mancini1994]
3. broadband squeezing [Kolobov1998]

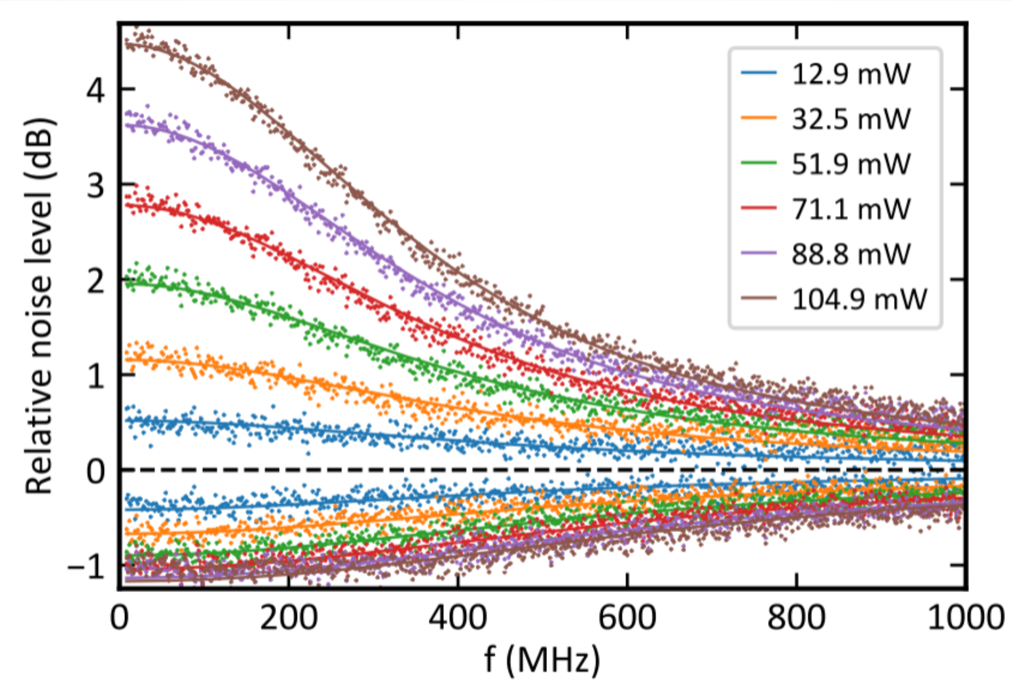


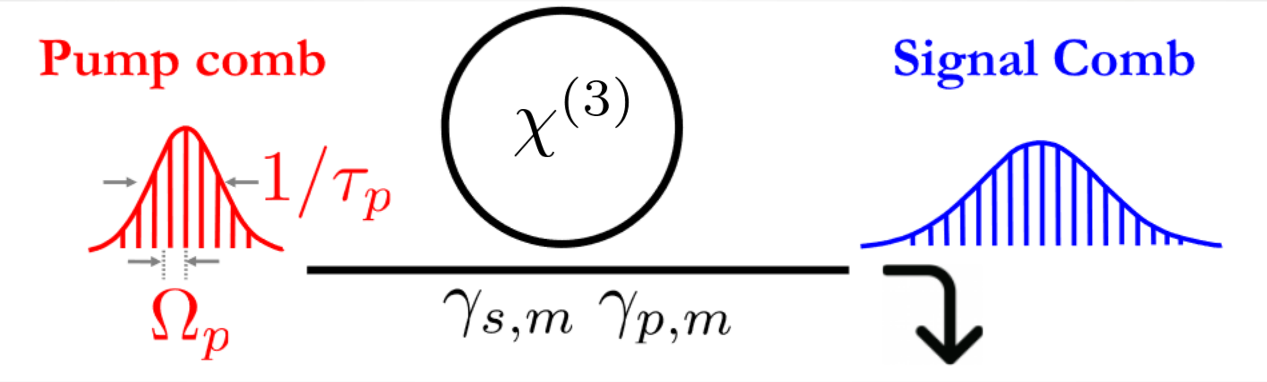
# Morphing supermodes

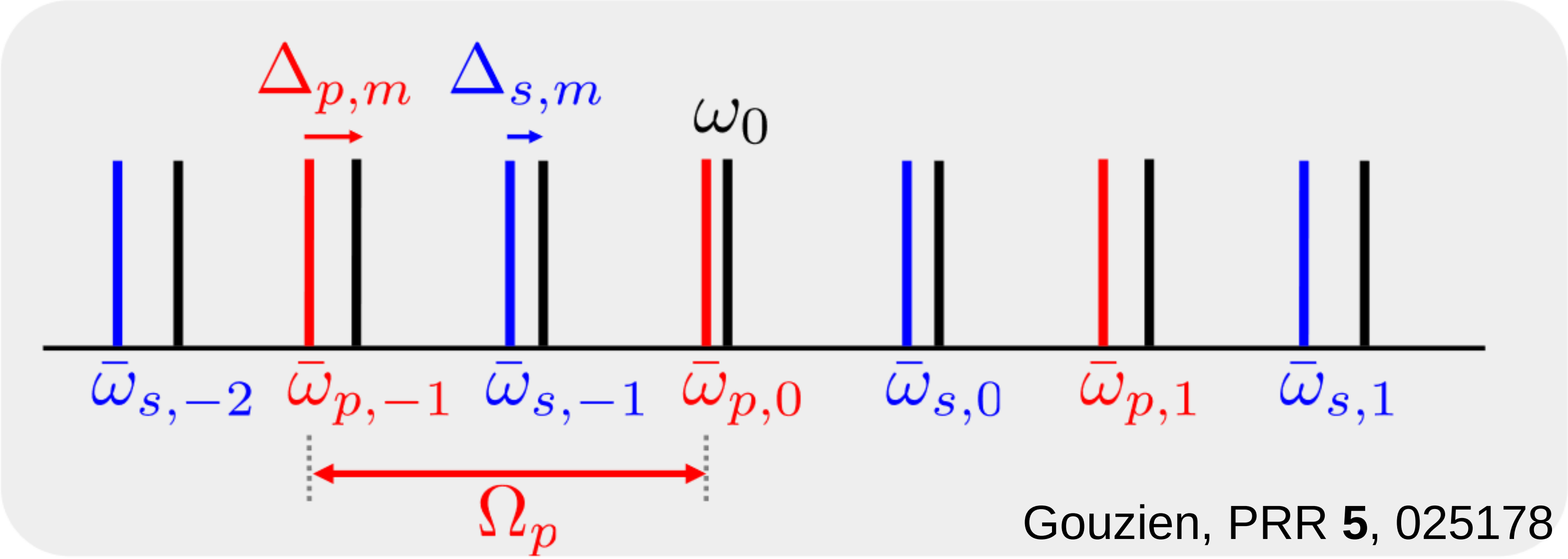
More recently

1. Xanadu broadband squeezing in microrings [Vaydia2020, Seifoory2022]
2. Advanced Ligo [Kimble2001,McCuller2020,Miller2023]



# micro-SPOPO

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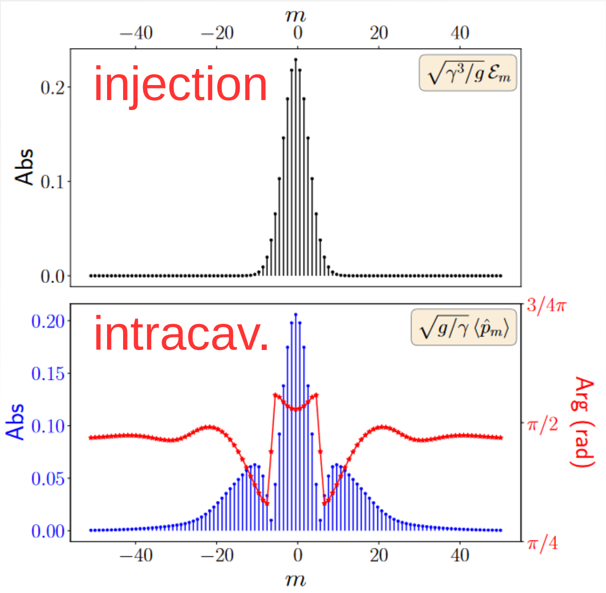
# Morphing supermodes

In µ-SPOPO

$$F_{m,n} = g \sum_l \langle \hat{p}_{m-l+n+1} \rangle \langle \hat{p}_l \rangle,$$

$$G_{m,n} = \Delta_{s,m} \delta_{[m-n]} + g \sum_l 2 \langle \hat{p}_{m+l-n} \rangle \langle \hat{p}_l \rangle^*,$$




1% below threshold

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$$F_{m,n} = g \sum_l \langle \hat{p}_{m-l+n+1} \rangle \langle \hat{p}_l \rangle,$$

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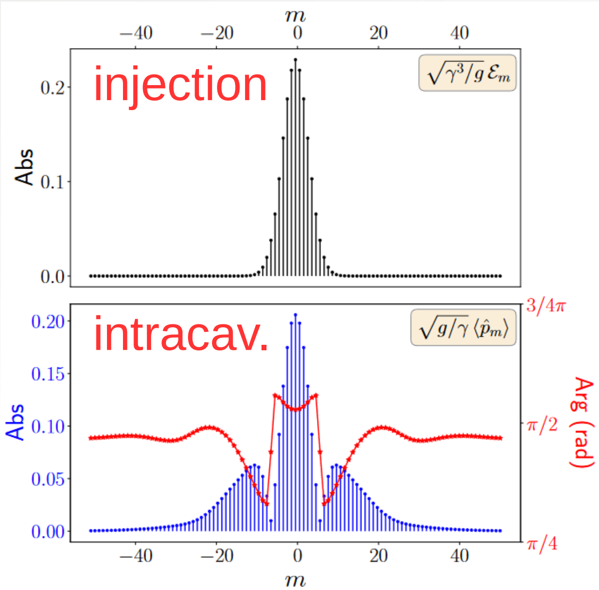


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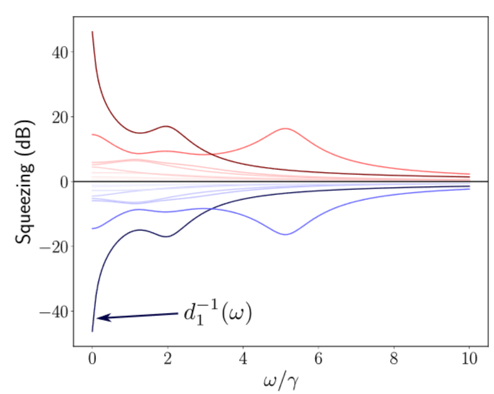


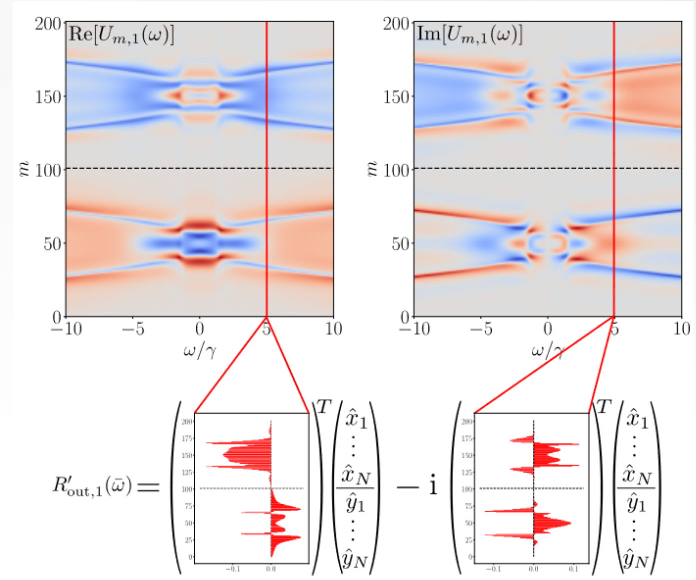


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# Interf. with memory effect

$$\underline{\hat{\mathrm{R}}}'_{\mathrm{out}}(\omega) = \boxed{S_{\mathrm{IME}}(\omega)}\,\underline{\hat{\mathrm{R}}}'_{\mathrm{in}}(\omega)$$

Transfer function

$$S_{\mathrm{IME}}(\omega) = \sqrt{2\Gamma_{\mathrm{IME}}}\left(\mathrm{i}\omega\mathbb{I} + \Gamma_{\mathrm{IME}} - \boxed{\mathcal{M}_{\mathrm{IME}}}\right)^{-1}\sqrt{2\Gamma_{\mathrm{IME}}} - \mathbb{I}$$

$$\mathcal{M}_{\mathrm{IME}} = \left(\begin{array}{c|c} \mathrm{Im}\left[G_{\mathrm{IME}}\right] & \mathrm{Re}\left[G_{\mathrm{IME}}\right] \\ \hline -\mathrm{Re}\left[G_{\mathrm{IME}}\right] & -\mathrm{Im}\left[G_{\mathrm{IME}}\right]^{\mathrm{T}} \end{array}\right)$$

$$\hat{H} = \hbar\sum_{m,n} G_{m,n}\hat{a}_m^{\dagger}\hat{a}_n$$

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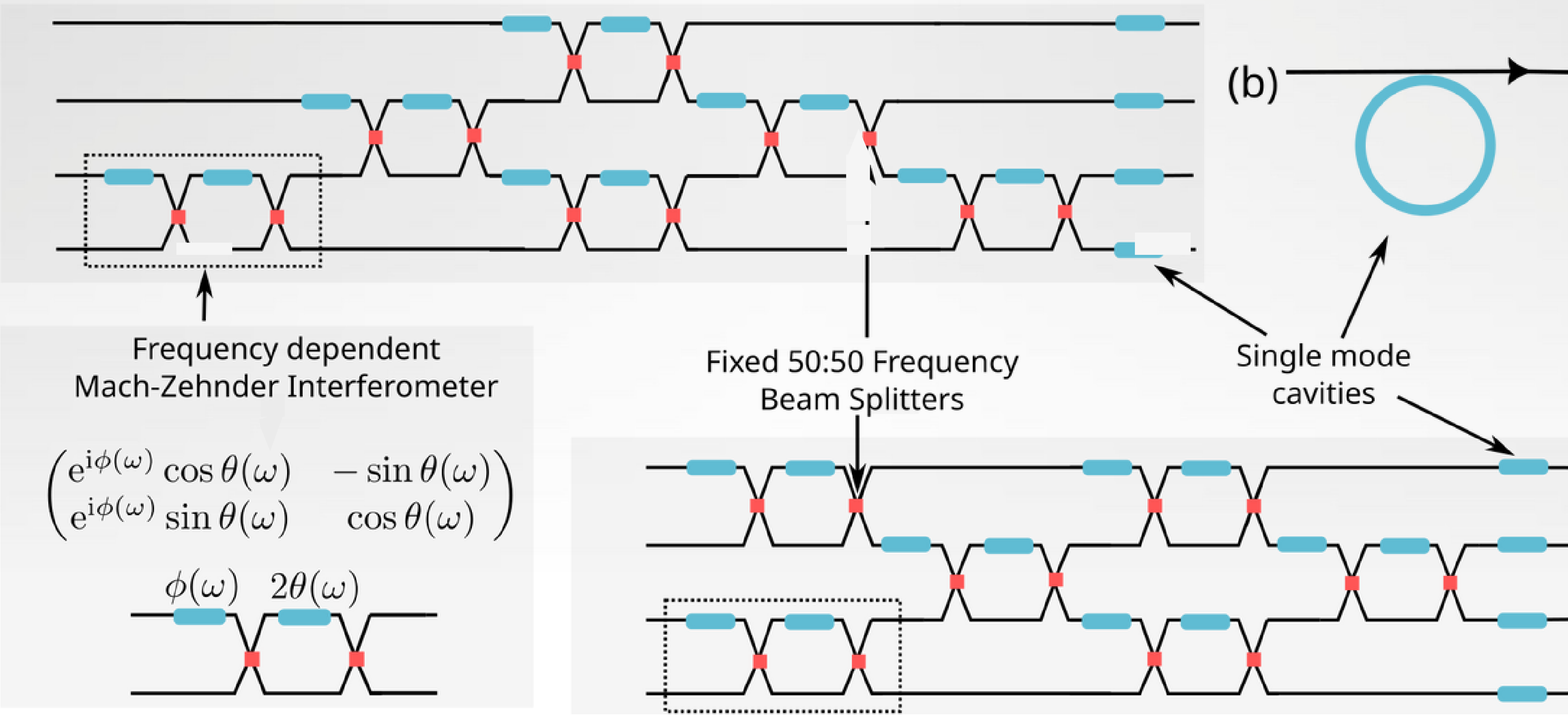
# Smooth one-mode decomp.

Triangular mesh




Rectangular mesh

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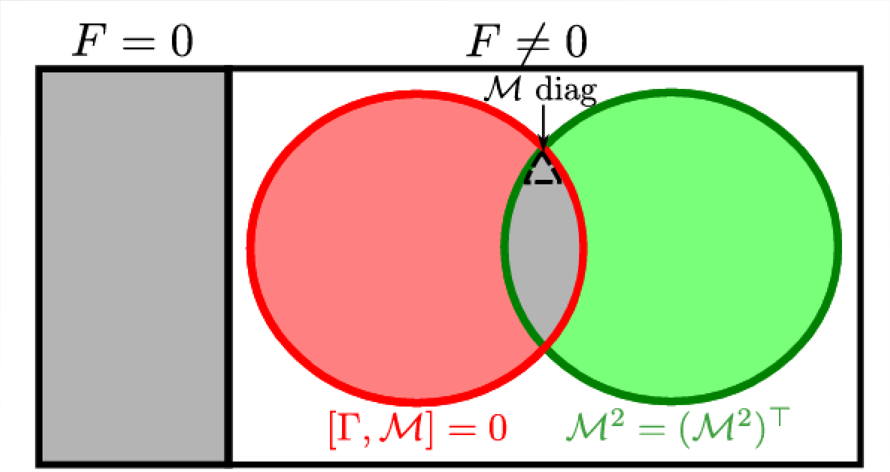
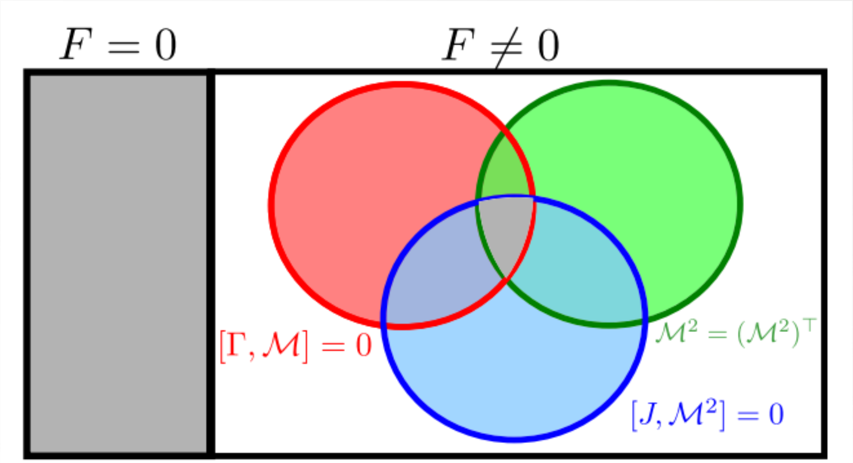
# Conditions to Hidden squeezing

With Vaccum Input




With Thermal Input

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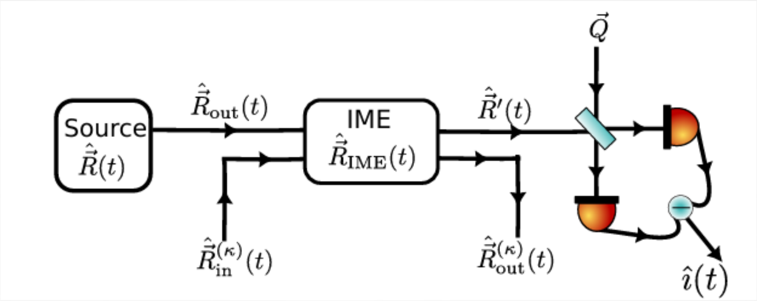
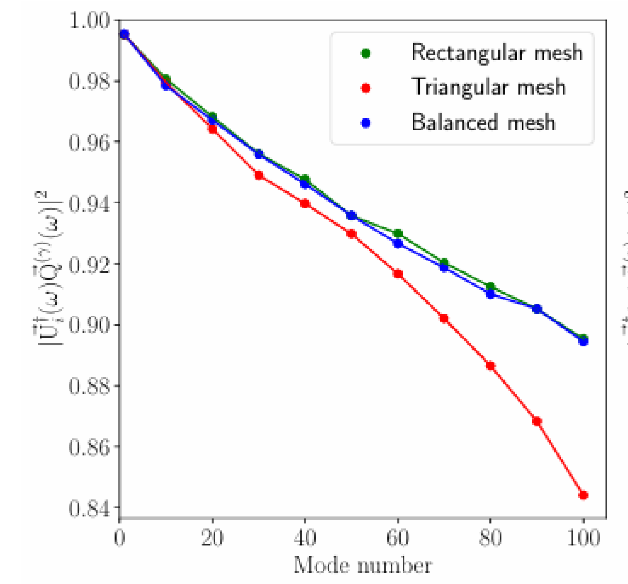
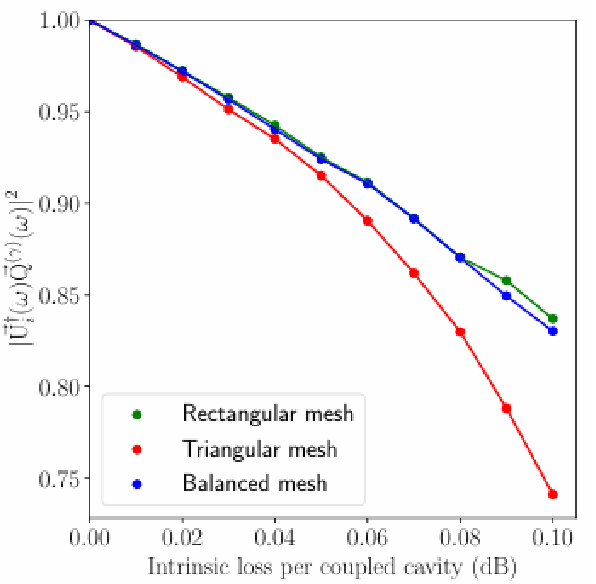
# IME with losses

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