Quantum Entropy of Black Holes

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Exact Quantum Entropy

Any black hole in ***any*** phase (= compactification) of the theory should be interpretable as an ensemble of quantum states ***including*** *finite size quantum gravity corrections.*

- ***Universal*** and ***extremely stringent*** constraint
- The famous Bekenstein-Hawking formula says that entropy is proportional to the area (in Planck units) for l**arge area.** The work of Strominger-Vafa gave its statistical interpretation for certain black holes.
- ***Finite area corrections*** connect to a broader problem of ***Quantum Holography at finite N****.*

References

Atish Dabholkar, João Gomes, Sameer Murthy,
Ashoke Sen, Don Zagier

Banerjee, Cabo-Bizet, Cardoso, Cheng, David, Denef, de Wit, Dijkgraaf, Gaiotto, Gibbons, Gupta, Guica, , Hardy, Hawking, Iliesiu, Jatkar, Jeon, Maldacena, Manschot, Mohaupt, Moore, Nekrasov, Nampuri, Ooguri, Pestun, Pioline, Ramanujan, Rademacher, Reys, Sen, Strominger, Shih, Troost, Turiaci, Yin, Verlinde, Verlinde, Vafa, Wald, Witten, Zagier, Zwegers, Turiaci, Iliesiu,…

Summary

- Striking progress over the past decade in computing exact quantum entropy of certain supersymmetric black holes including all perturbative and non-perturbative corrections to the Bekenstein-Hawking entropy.
- The resulting entropy is a logarithm of an integer in precise agreement with integral microscopic degeneracies.
- Nontrivial interplay between topics in number theory and topology like modular forms, Rademacher expansion, Kloosterman sums, Chern-Simons theory and methods in physics like localization in supergravity.
- Connections with recent developments like the SYK

Some Questions

1. What is the quantum generalization of Bekenstein-Hawking formula for black holes with finite area in Planck unit?
2. Does it have a statistical interpretation beyond the leading answer?
3. What is the ensemble in which it is defined?
4. Is it calculable including both perturbative and nonperturbative corrections?
5. Is AdS/CFT holography valid at finite N?

Quantum Entropy and Holography

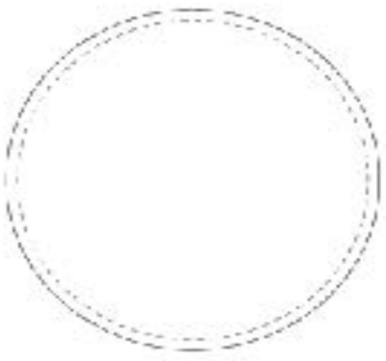
- Near horizon of supersymmetric BPS black holes in 4d has an $AdS_2 \times S^2$ factor

$$ds^2 = l_*^2 \left[(r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} + d\psi^2 + \sin^2 \psi d\phi^2 \right]$$

$$F^I = -\, i\, e_*^I dr \wedge d\theta + P^I \sin \psi\, d\psi \wedge d\phi \,, \qquad X_*^I$$

- One can apply usual rules of holographic correspondence keeping in mind some of the important peculiarities of AdS_2

AdS_2/CFT_1 Holography



- Bulk AdS_2 is Poincaré disk. Put a cutoff at $r = r_0$
- Boundary CFT_1 is a finite dimensional Hilbert space of dimension $d(Q,P)$ and zero Hamiltonian $H=0.$

$$Z_{CFT}(Q,P) := \text{Tr}\, e^{(-2\pi r_0 H)} = d(Q,P)$$

Quantum Entropy

- Exponential of quantum entropy is given by *Sen (09)*

$$W(Q,P) := \langle \exp\left(-\frac{i}{2} Q_I \int A^I\right) \rangle_{ren}$$

- Path integral ***over all string fields*** *with an insertion of a Wilson line and with appropriate boundary conditions and renormalization.*
- Reduces to Bekenstein-Hawking-Wald for large *(Q, P)*
- Quantum holography requires

$$W(Q,P) = d(Q,P)$$

W(Q, P)	d(Q, P)
Black Hole (Q, P)	Brane (Q, P)
Quantum Entropy	Counting of States
AdS_2	CFT_1
Quantum Geometry	Hilbert Space

A quantum generalization of

Bekenstein-Hawking ***Boltzmann***

Can we compute both sides?



Defining W(Q, P)	AdS/CFT
Counting d(Q, P)	D-branes, Duality
Degeneracy	Index, Modularity
Path integral	Localization
Sugra Localization	Off-Shell Sugra
Sugra Action	Nonrenormalization
Kloosterman phases	Chern-Simons terms
Wall-crossing	Mock Jacobi Forms

Choice of Ensemble

The gauge field behaves as

$$A_\theta^I \sim -ie^I r + \mu^I; \qquad F_{r\theta} = -ie^I$$

A natural choice is to hold the growing mode fixed. Thus hold the electric field (and hence the charge) fixed and let the chemical potential fluctuate.

Contrast this with the higher dimensional case.

Corresponds to *microcanonical* ensemble.

Degeneracy = Index

- The near horizon AdS_2 has *SU(1, 1)* symmetry.
- With four supersymmetries, the closure of algebra implies *SU(1, 1|2)* supergroup as symmetry.
- Hence, horizon must have *SU(2)* symmetry. Consistent with the fact that supersymmetric black holes in 4d are spherically symmetric.
- **Microcanonical** ensemble implies ***J = 0***.
- All horizon degrees of freedom are bosonic!

$$\mathbf{Tr}(\mathbf{1}) = N_B + N_F = \mathbf{Tr}(\mathbf{-1})^F = N_B - N_F$$

Dyonic States in N=8 theory

Type II on T^6. Dyons with charge vector *(Q, P)*

Duality invariant $\Delta = Q^2 P^2 - (Q \cdot P)^2$

$$Z(\tau, z) = \prod_{r}^{\infty} \frac{(1-q^r y)^2 (1-q^r y^{-1})^2}{(1-q^r)^4} \qquad (y := e^{2\pi i z})$$

$$= \sum_{n=-1}^{\infty} c(n,l) q^n y^l \quad ; \quad c(n,l) = C(4n - l^2)$$

$$d(\Delta) = (-1)^{\Delta+1} C(\Delta)$$

$$W(\Delta) = d(\Delta)?$$

To attempt such a comparison it's useful to use

Hardy-Ramanujan-Rademacher expansion

Exact generalization of Cardy formula

$$d(\Delta) = \sum_{c=1}^{\infty} c^{-9/2} \tilde{I}_{7/2}\left(\frac{\pi}{c}\sqrt{\Delta}\right) K_c(\Delta)$$

The c=1 Bessel function sums all perturbative corrections to entropy. The c>1 are non-perturbative

$$\tilde{I}_{7/2}(z) \sim \exp[z - 2\log z + \frac{c}{z} + \cdots]$$

Computing $W(\Delta)$

- The structure of the microscopic answer suggests that $W(\Delta)$ should have an expansion

$$W(\Delta) = \sum_{c=1}^{\infty} W_c(\Delta)$$

- We will find that $W_c(\Delta)$ arises from an $\mathbb{Z}_c$ orbifold saddle point of the path integral.
- The higher *c* are exponentially subleading. Unless one can evaluate each of them *exactly* it is not particularly meaningful to add them.

Equivariant Localization in supergravity enables us to do this.

Modular Symmetry

- A holomorphic function $F(\tau)$ on the upper half complex plane is a modular form of weight k, if it transforms as

$$F(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k F(\tau)$$

for *a, b, c, d, k* integers and *ad-bc =1*

The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *form the group SL(2, Z) under matrix multiplication.* ***Highly Symmetric.***

Jacobi forms

A Jacobi form of weight k and index m

$$\varphi_{k,m}(\tau, z)$$

`modular' with weight k

`elliptic' in z with index m

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z)$$

$$(\forall \quad \lambda, \mu \in \mathbb{Z})$$

Our $Z(\tau, z)$ is a Jacobi form of weight 2 index 1.

Elliptic and modular properties again imply a bit more involved Hardy-Ramanujan-Rademacher expansion:

$$C(\Delta) = N \sum_{c=1}^{\infty} c^{-9/2} \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right) K_c(\Delta)$$

Exact degeneracies are known also for dyonic states in N=4 in terms of Fourier coefficients of Siegel modular forms and exhibit an intricate structure of wall-crossings in the moduli space. They connect to the mathematics of Mock modular forms introduced by Ramanujan a century ago. More general N=2 degeneracies relate to Donaldson-Thomas invariants.

Generalized Kloosterman Sum $K_c(\Delta)$

$$\sum_{\substack{-c\le d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\; M^{-1}(\gamma_{c,d})_{\nu 1}\; e^{2\pi i \frac{a}{c}(-1/4)}$$

$$\nu = \Delta \mod 2$$

$$M^{-1}(\gamma)_{\nu\mu} = C\sum_{\epsilon=\pm}\sum_{n=0}^{c-1}\epsilon\, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2-2(\nu+1)(2rn+\epsilon(\mu+1))+a(2rn+\epsilon(\mu+1))^2\right]}$$

*Intricate number theoretic phases, highly subleading in large area expansion, but essential for **integrality.** Quantum holography requires that the bulk must reproduce these nonperturbative phases.* ***And it does!***

Power of Quantum Holography (*AdS_2* /CFT$_1$)

The path integral *W(Δ)* in the near horizon *AdS_2* can be defined as a generalization of Wald entropy.

Includes nonlocal effects from massless loops.

Quantum Holography implies two things:

1. *W(Δ)* = *d(Δ)* (nontrivial prediction for a path integral)

   Path integral must be an integer!

2. *d(Δ)* = *W(Δ)* (nontrivial prediction for an index)

   Index must be positive! (index = ***degeneracy.*****)**

Localization of Path Integrals

We are interested in a path integral of the form

$$Z = \int_{\mathscr{M}} d\mu \, e^{S_{ren}}$$

with a supersymmetric measure and action.

Localization techniques make it possible to evaluate such integrals. We have learnt a great deal about the noperturbative structure of QFT which was otherwise inaccessible without localization.

Duistermaat-Heckmann(82)..Witten (88) Nekrasov (02) Pestun (04) …

Can we localize **supergravity** path integrals?

Localization

Consider a supermanifold $\mathscr{M}$ with an integration measure $d\mu$. Let Q be an odd (fermionic) vector field on this manifold that satisfies:

1. $Q^2 = H$ for a compact bosonic vector field H.
2. $div_\mu(Q) = 0$ i. e. the measure is invariant under Q.

Note that Q is nilpotent on H-invariant configurations

Allows one to study `**Equivariant** Cohomology'

Field space is an (infinite-dimensional) supermanifold
Q is a supersymmetry, *H* is a Killing field.

Deformation Invariance

- Consider a deformation of the original integral

$$Z(\lambda) = \int_{\mathcal{M}} d\mu \; e^{S_{ren} - \lambda QV}$$

where V is an *H*-invariant fermionic function

$$H(V) = Q^2(V) = 0;$$

- One can then prove easily that

$$\frac{d}{d\lambda} Z(\lambda) = \int_{\mathcal{M}} d\mu \, Q(V \, e^{S_{ren} - \lambda QV}) = 0$$

Beauty of Off-Shell Supergravity

1. Supersymmetry transformations are written down once and for all (much like coordinate transformations) **independent of the action.**

– Algebra closes without using equations of motion.

– Essential for using SUSY inside a path integral.

2. It nicely separates the problem into two parts.

– Find the offshell localizing solutions once and for all independent of the physical action.

– Evaluate the renormalized action on the localizing manifold for any given compactification.

Challenges in Supergravity

- In Euclidean gravity the conformal factor has a wrong sign kinetic term.
- Since metric is dynamical what does it mean to have a background with a symmetry?
- At a more fundamental level, since all symmetries such as *Q* and *H* are ***gauge*** symmetries, how can we even get started with localization?
- Unlike in QFT, the action has higher derivative terms and is nonrenormalizable. In particular, there are infinite number of terms in the effective action.

Strategy

- Use background field BRST quantization:
  field = background field + quantum field.
- Gauge parameters that don't vanish at infinity generate the Killing symmetries of the background. Use these symmetries to localize.
- Use off-shell superconformal supergravity.
- Nonrenormalization theorem: nonchiral D-terms don't contribute. Huge simplification.
- A single prepotential ***F*** specifies the chiral terms. Moreover receives no quantum corrections.

Off-shell Localizing Solutions

- One finds ***off-shell*** localizing instantons in AdS_2 for supergravity coupled to n_v vector multiplets with scalars X^I and auxiliary fields $Y^I_{12} = Y^I_{21} := Y^I$

$$X^I = X^I_* + \frac{C^I}{r}, \qquad Y^I = \frac{2C^I}{r}, \qquad C^I \in R\,; \ (I = 0, 1, \ldots n_v)$$

These solutions are ***universal*** in that they are *independent of the physical action* and follow entirely from the off-shell susy transformations.

Valid for any physical action.

Supergravity Action

$$
\begin{aligned}
& 8\pi e^{-1}\mathcal{L} \\
= \quad & (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2}R) \\
+ \quad & \big[i\nabla_\mu F_I \nabla^\mu \bar{X}^I \\
+ \quad & \frac{1}{4} i F_{IJ} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij})(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{ij}_{ab}\,\varepsilon_{ij}) \\
- \quad & \frac{1}{8} i F_I (F^{+I}_{ab} - \frac{1}{4} X^I T_{abij}\,\varepsilon^{ij}) T^{ij}_{ab}\,\varepsilon_{ij} \\
- \quad & \frac{1}{8} i F_{IJ} Y^I_{ij} Y^{Jij} - \frac{i}{32} F\,(T_{abij}\,\varepsilon^{ij})^2 \\
+ \quad & \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik}\varepsilon^{jl}\widehat{B}_{ij}\widehat{B}_{kl} - 2\widehat{F}^-_{ab}\widehat{F}^-_{ab}) \\
+ \quad & \frac{1}{2} i \widehat{F}^{-ab} F_{\widehat{A}I} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij}) - \frac{1}{4} i \widehat{B}_{ij} F_{\widehat{A}I} Y^{Iij} + \text{h.c.} \big] \\
- \quad & i(X^I \bar{F}_I - F_I \bar{X}^I) \cdot (\nabla^a V_a - \frac{1}{2} V^a V_a - \frac{1}{4}|M_{ij}|^2 + D^a \Phi^i{}_\alpha D_a \Phi^\alpha{}_i)\ .
\end{aligned}
$$

Renormalized Action

The renormalized action for prepotential ***F*** simplifies:

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \bar{F}\Big(\frac{\phi^I - ip^I}{2}\Big) \right]$$

$$\phi^I := e_*^I + C^I$$

$\frac{1}{2}(\phi^I + ip^I)$ is the off-shell value of X^I at the origin of the Poincaré disk.

Dabholkar Gomes Murthy (11, 13)

Final Answer

$$W_1(Q,P) = \int [d\phi]\, e^{-\pi Q_I \phi^I + Im F(\phi + iP)}\, Z_{det}(\phi)\, Z_{inst}$$

$$Z_{det}(\phi) = \exp\left[-K(\phi + iP)\left(n_v - n_h + 23/12\right)\right]$$

$$e^{-K} := -\, i(X^I \bar{F}_I - \bar{X}^I F_I)$$

A finite dimensional integral determined entirely in terms of the prepotential (+ possibly point instantons).

Final integral

The prepotential for the truncated theory is

$$F(X) = -\frac{1}{2}\frac{X^1}{X^0}\sum_{a,b=2}^{7} C_{ab}X^a X^b \qquad (n_v = 7)$$

(dropping the extra gravitini multiplets of ***N=8***)

The *path* integral reduces to the Bessel integral (more complete justification later)

$$W_1(\Delta) = N\int \frac{ds}{s^{9/2}}\exp\left[s + \frac{\pi^2\Delta}{4s}\right]$$

$$W_1(\Delta) = \tilde{I}_{7/2}(\pi\sqrt{\Delta})$$

Orbifold Contributions

- Including the M-theory circle, there is a family of geometries $\mathcal{M}_{c,d}$ that are asymptotically $AdS_2 \times S^1$:

$$ds^2 = (r^2 - \frac{1}{c^2})d\theta^2 + \frac{dr^2}{r^2 - \frac{1}{c^2}} + R^2 \left(dy - \frac{i}{R}(r - \frac{1}{c})d\theta + \frac{d}{c}d\theta \right)^2$$

- Freely acting $\mathbb{Z}_c$ orbifolds of BTZ black hole.

  Related to the $SL(2,\mathbb{Z})$ family in AdS_3

Localization justifies keeping these subleading saddles.

Subleading Bessel Functions

- Contributions from these smooth orbifolds explain the Bessel functions **for all *c*** with correct argument because for each orbifold the localized solutions are the same but the renormalized action is reduced by a factor of *c*
- *What about the Kloosterman sums?*

How can a SUGRA path integral possibly reproduce this intricate number theoretic structure ?

Generalized Kloosterman Sum $K_c(\Delta)$

$$\sum_{\substack{-c\leq d<0;\\(d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\; M^{-1}(\gamma_{c,d})_{\nu 1}\; e^{2\pi i \frac{a}{c}(-1/4)}$$

$$\nu = \Delta \mod 2$$

$$M^{-1}(\gamma)_{\nu\mu} = C \sum_{\epsilon=\pm}\sum_{n=0}^{c-1} \epsilon\, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2-2(\nu+1)(2rn+\epsilon(\mu+1))+a(2rn+\epsilon(\mu+1))^2\right]}$$

*Number theoretic phases essential for **integrality***

Chern-Simons-Witten Theory

- Our localization analysis so far ignored the topology.
- The Chern-Simons terms in the bulk and the boundary terms are sensitive to the global properties of $\mathcal{M}_{c,d}$
- Additional saddles specified by holonomies of flat connections. Various phases from CS terms assemble nontrivially into the Kloosterman sum.
- Closely related to knot invariants of Lens space $\mathcal{L}_{c,d}$ using the surgery formula.

Kloosterman and Chern-Simons

$$I(A) = \int_{\mathcal{M}_{c,d}} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A^3\right)$$

In our problem we have three relevant groups

$$U(1)^{n_v+1} \qquad SU(2)_L \qquad SU(2)_R$$

$$\sum_{\substack{-c\leq d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)} \, M^{-1}(\gamma_{c,d})_{\nu 1} \, e^{2\pi i \frac{a}{c}(-1/4)}$$

Dabholkar Murthy Gomes (14)

Multiplier System from $SU(2)_L$

There is an explicit representation of the Multiplier matrices that is suitable for our purposes.

$$M^{-1}(\gamma)_{\nu\mu} = C \sum_{\epsilon=\pm} \sum_{n=0}^{c-1} \epsilon \, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2 - 2(\nu+1)(2rn+\epsilon(\mu+1)) + a(2rn+\epsilon(\mu+1))^2\right]}$$

Unlike $SU(2)_R$ the holonomies of $SU(2)_L$ are not constrained by supersymmetry and have to be summed over which gives precisely this matrix.

(Assuming usual shift of k going to k +2)

Remarkably AdS path integral reproduces all details.

A path integral (a complex analytic continuous object)

yields an integer (a number theoretic discrete object).

$W(\Delta)$ = integer!

An *IR* Window into the *UV*

- It counts with precision *nonperturbative* states with masses much higher than the string scale.
- If we did not know the spectrum of branes a priori we could in principle deduce it. e.g. in *N=6* models!

Cardy goes to Hardy!

We defined the quantum entropy $W(\Delta)$ as a path integral of N=8 supergravity fields on $AdS_2 \times S^2$

$$W(\Delta) \sim \exp[\frac{A(\Delta)}{4}] = \exp[\pi\sqrt{\Delta}] \sim d(\Delta)\,, \qquad \Delta \gg 1$$

Classical Bekenstein-Hawking Entropy = Cardy

Exact Quantum Entropy = Hardy

It's remarkable and highly nontrivial that ***all*** quantum corrections are computable in string theory which combine into a specific integer.

Localization in Supergravity

Localization in gauge theory has some subtleties since susy algebra closes up to gauge transformations. Problem is exacerbated in sugra.

- The `structure constants' of supergravity gauge algebra are field-dependent: soft algebra
- The metric is dynamical.

Earlier we dealt with them heuristically. One can set up a Background Field BRST formalism to deal with both these problems in a systematic way by deforming the BRST Q into equivariant Q.

de Wit Murthy Reys; Jeon Murthy

Off-shell N=8 supergravity

- Off-shell formulation of the full N=8 algebra requires infinite number of auxiliary fields.
- We used an N=2 truncation which was a good start for finding the localizing instantons but is is not satisfactory for computing the determinants.
- It is sufficient to realize only two super symmetries off-shell but on the entire supermultiplet of N=8

(*Reys-Murthy; Bergeshoeff-de Roo-de Witt; Ciceri-Sahoo; Ilesieu-Murthy-Turiaci)*

Treatment of the Infinity of Zero Modes

- Zero modes of the quadratic fluctuation operator.
- Arbitrary fluctuations of the boundary of the metric corresponding to $Diff(S^1)$ or one copy of Virasoro. except the three dimensional global $SL(2,R)$ Killing symmetries generated by $\{L_0, L_1, L_{-1}\}$.
- The action for these zero modes is precisely given by the Schwarzian action and the integral over the zero modes gives a volume factor.
- We need to evaluate the volume $Vol_0(Q,T)$ as a function of charges and the temperature.

Relation to SYK

- The physics of a near-extremal black hole in a range of parameters (temperature and chemical potential) is fully captured by these Camporesi-Higushi `graviton zero modes' with Schwarzian action related to SYK model (*Moitra, Sake, Trivedi*)
- Because of the three bosonic zero modes,

$$Vol_0(Q,T) = (Q^3T)^{3/2}$$

- The volume goes to zero in the extremal limit and particular temperature dependence is precisely what is obtained in the SYK model.

Supersymmetric Black Holes

- Fortunately, for supersymmetric black holes we also have to deal with three (complex) fermionic zero modes which precisely cancel this temperature dependence with a finite answer in the zero temperature limit.
- The volume of zero modes also picks up factor of `*c* ‘ in the orbifold limit neatly explaining all factors.

(Ilesieu, Murthy, Turiaci)

$$d(\Delta) = \sum_{c=1}^{\infty} c^{-9/2}\, \tilde{I}_{7/2}\,(\frac{\pi}{c}\sqrt{\Delta})\, K_c(\Delta)$$