Neutrino Mixing from Modular Flavor Symmetries

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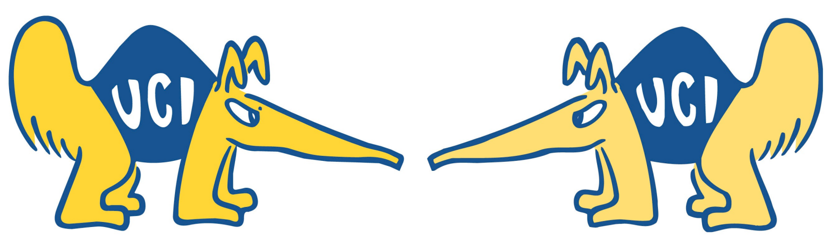


PASCOS2024, ICISE, Quy Nhon, Vietnam, July 8, 2024

Where Do We Stand?

NuFIT 5.2 (2022)

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

- ➡ hints of $\theta_{23} \neq \pi/4$
- ➡ expectation of Dirac CP phase δ

Recent T2K-NOvA joint analysis: (Z. Vallari, FNAL, Feb'24)

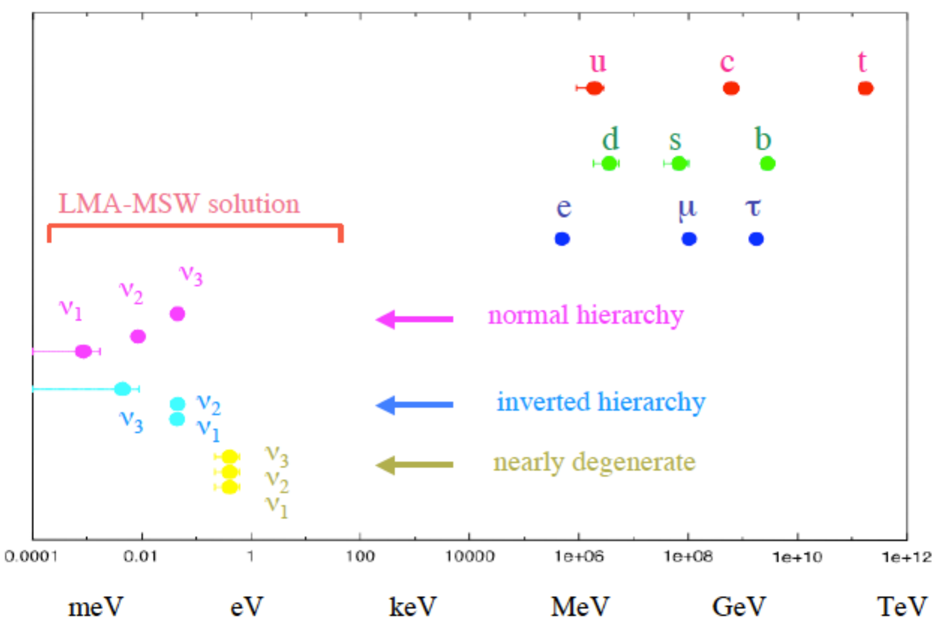
slight preference for IO; $\delta \simeq -\pi/2$; $\theta_{23} > 45^o$

T2K-NOvA-DayaBay ⇒ NO

Open Questions - Theoretical

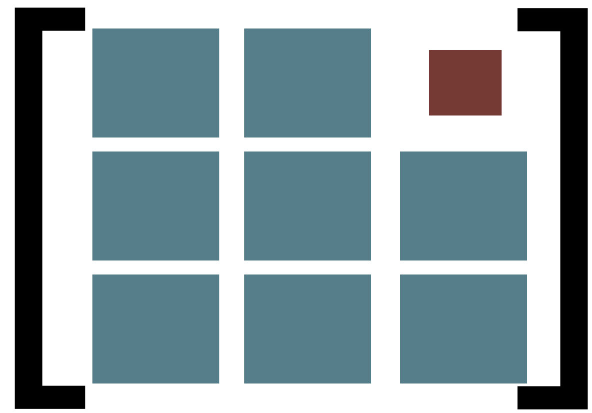
☞ **Smallness of neutrino mass:**

$$m_\nu \ll m_{e,\,u,\,d}$$

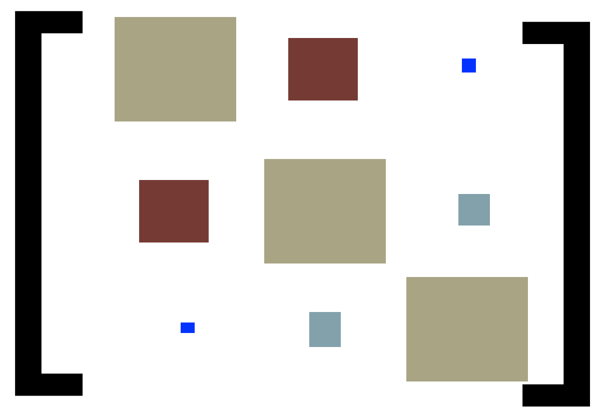


Fermion mass and hierarchy problem ➟ Dominant fraction (22 out of 28) of free parameters in SM

☞ **Flavor structure:**

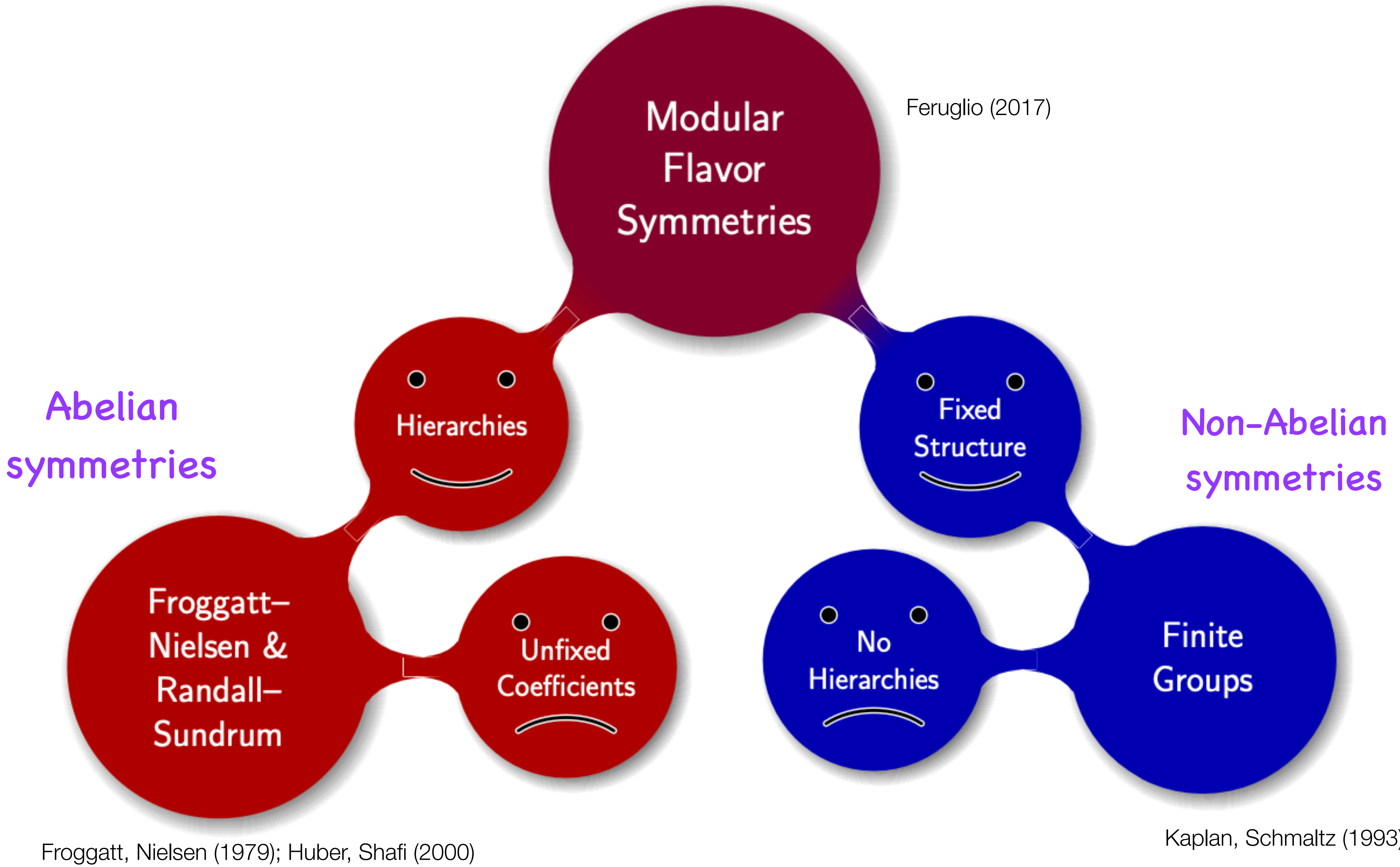


leptonic mixing

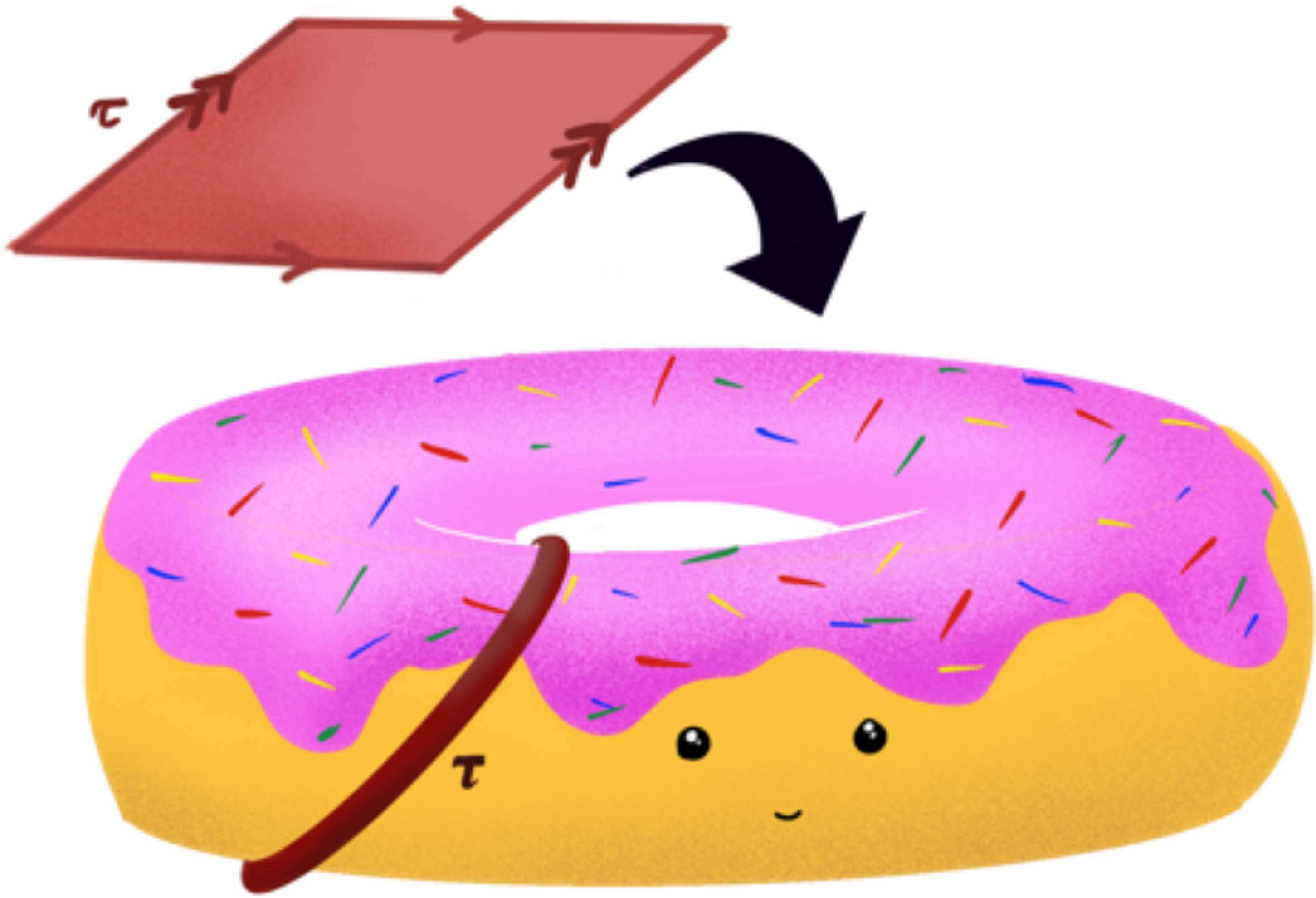


quark mixing

Theories of Flavor



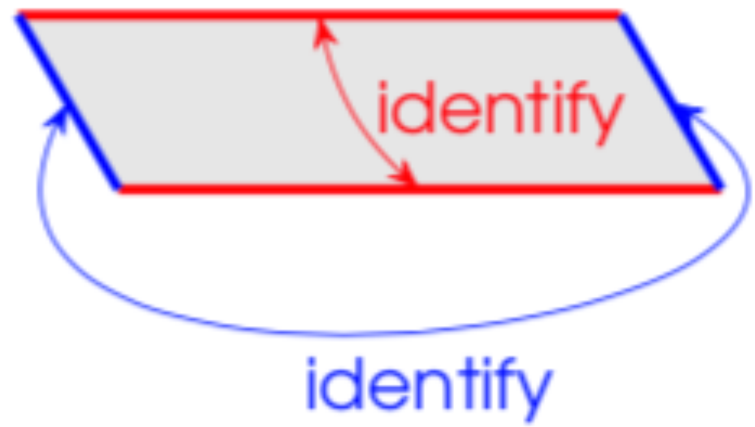
Modular Flavor Symmetries

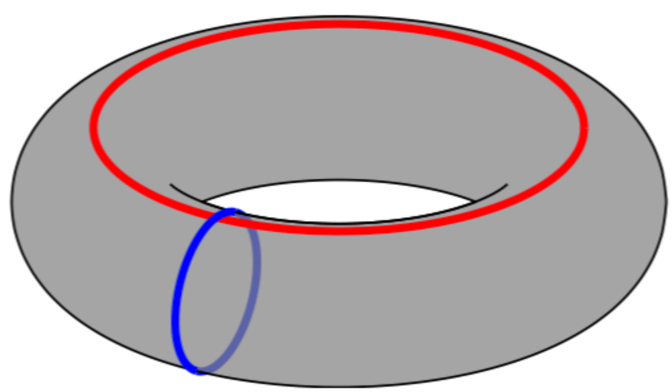


Artwork by Shreya Shukla

Donuts = TORI

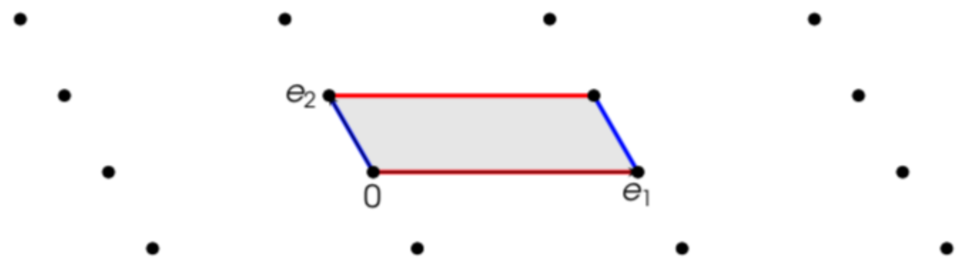
constructed from parallelogram





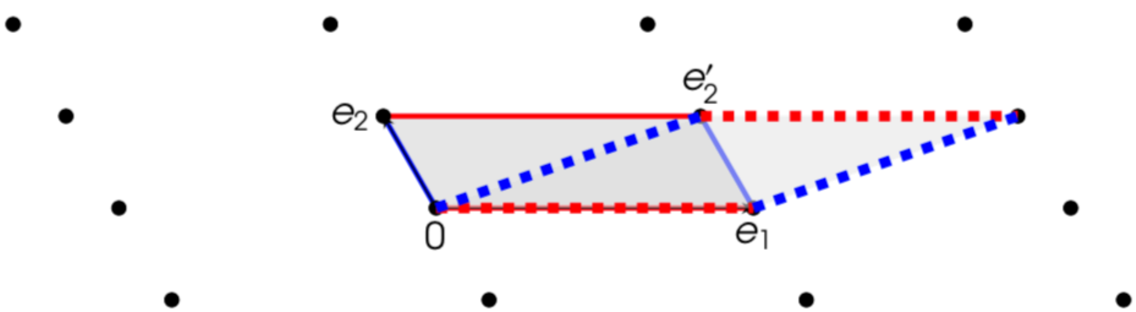
two cycles

Modular Symmetries



edges ⇒ lattice basis vectors

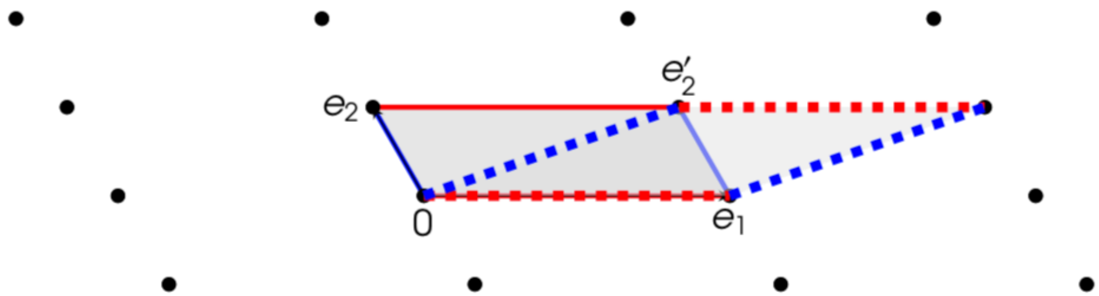
points in plane identified if differ by a lattice translation



Equivalent TORI related by Modular Symmetries

Modular Symmetries

- TORI: fundamental domain not unique



- Basis Vectors are related: $\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \overset{\gamma}{\mapsto} \begin{pmatrix} e'_2 \\ e'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} =: \gamma \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$

$a, b, c, d \in \mathbb{Z}$

- Volume of fundamental domain the same $\Rightarrow$ $\det \gamma = 1$

Modular Symmetries

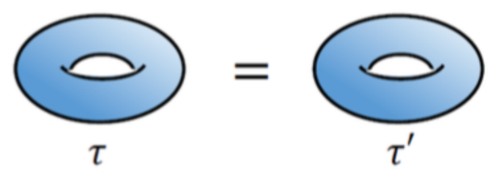
- Two basic transformations:

$$T\ :\ e_2 \mapsto e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S\ :\ e_1 \mapsto e_1' = e_2 \quad \text{and} \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

- In complex coordinates: modulus $\tau = e_2/e_1$

$$\tau \overset{S}{\longmapsto} \frac{-1}{\tau} \quad \text{and} \quad \tau \overset{T}{\longmapsto} \tau + 1$$



- S and T generate $\mathrm{SL}(2,\mathbb{Z})$ and satisfy

$$\boxed{S^2 = (S\,T)^3 = \mathbb{1}}$$

Modular Symmetries

- **Finite Modular Group** (quotient group): $\Gamma_N := \Gamma/\Gamma(N)$ where principal congruence group Γ(N) is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2 \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

Γ

- Generators of the quotient group Γ_N satisfy

$$S^2 = 1, \quad (ST)^3 = 1, \quad T^N = 1$$

- Some examples

$$\Gamma_2 \simeq S_3, \quad \Gamma_3 \simeq A_4, \quad \Gamma_4 \simeq S_4, \quad \Gamma_5 \simeq A_5$$

Modular Symmetries

Feruglio (2017)

- Imposing modular symmetry Γ on the Lagrangian:

$$\mathscr{L} \supset \sum Y_{i_1,\, i_2,\, \ldots\ldots\, i_n} \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_n}$$

$$\tau \xrightarrow{\gamma} \gamma\tau := \frac{a\tau + b}{c\tau + d}\,,$$

$$\Phi_j \xrightarrow{\gamma} (c\tau + d)^{k_j} \rho_{r_j}(\gamma) \Phi_j\,, \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

k_i : integers

representation matrix of Γ_N

- Yukawa Couplings = Modular Forms at level "N" w/ weight "k"

$$f_i(\gamma\tau) = (c\tau + d)^{-k} [\rho_N(\gamma)]_{ij} f_j(\tau)$$

$k = k_{i_1} + k_{i_2} + \ldots + k_{i_n}$

representation matrix of Γ_N

A Toy Modular A_4 Model

Feruglio (2017)

- Weinberg Operator $\mathcal{W}_\nu = \frac{1}{\Lambda}[(H_u \cdot L)\, Y\, (H_u \cdot L)]_1$
- Traditional A4 Flavor Symmetry
  - Yukawa Coupling Y $\rightarrow$ Flavon VEVs (A_4 triplet, 6 real parameters)

$$Y \to \langle \phi \rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Longrightarrow \quad m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2a & -c & -b \\ -c & 2b & -a \\ -b & -a & 2c \end{pmatrix}$$

- Modular A4 Flavor Symmetry
  - Yukawa Coupling Y $\rightarrow$ Modular Forms (A4 triplet, 2 real parameters)

$$Y \to \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} \quad \Longrightarrow \quad m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Modular Forms

Feruglio (2017)

- Known mathematical functions:
  - Level (N) = 3, Weight (k) = 2

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} X_2^2(\tau) \\ \sqrt{2}X_1(\tau)X_2(\tau) \\ -X_1^2(\tau) \end{pmatrix}$$

$$X_1(\tau) = 3\sqrt{2}\frac{\eta^3(3\tau)}{\eta(\tau)}$$

$$X_2(\tau) = -3\frac{\eta^3(3\tau)}{\eta(\tau)} - \frac{\eta^3(\tau/3)}{\eta(\tau)}$$

Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

A Toy Modular A_4 Model

Feruglio (2017)

- Input Parameters:

$$\tau = 0.0111 + 0.9946i \qquad v_u^2/\Lambda$$

3 free parameters ⇒

3 mass,
3 angles,
3 CP phases

- Predictions: inverted ordering

$$\frac{\Delta m^2_{sol}}{|\Delta m^2_{atm}|} = 0.0292$$

$$\sin^2\theta_{12} = 0.295 \qquad \sin^2\theta_{13} = 0.0447 \qquad \sin^2\theta_{23} = 0.651$$

$$\frac{\delta_{CP}}{\pi} = 1.55 \qquad \frac{\alpha_{21}}{\pi} = 0.22 \qquad \frac{\alpha_{31}}{\pi} = 1.80 \quad .$$

$$m_1 = 4.998 \times 10^{-2}\ eV \qquad m_2 = 5.071 \times 10^{-2}\ eV \qquad m_3 = 7.338 \times 10^{-4}\ eV$$

Predictive Power of Modular Symmetries

- Ingredients
  - Modular invariance
  - Holomorphy
  - Finiteness
- However, typical observables are not holomorphic, e.g.

$$\mathscr{W} = \frac{\mathscr{M}(\tau)}{2}\Phi^2$$

$$\mathscr{K} = \frac{1}{(-i\tau + i\bar{\tau})^{k_\Phi}}\overline{\Phi}\Phi$$

$\Rightarrow$

$$m_{physical} = m_{physical}(\bar{\tau}, \tau) = |\mathscr{M}(\tau)|(-i\tau + i\bar{\tau})^{k_\Phi}$$

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$\Rightarrow$

$$m_{physical} = m_{physical}(\bar{\tau}, \tau)$$

$$= |\mathscr{M}(\tau)|(-i\tau + i\bar{\tau})^{k_\Phi}$$

Are there observables fulfilling the three properties?

Holomorphic Observables

- Typical model

$$\mathscr{W}\,|_{lepton} = Y_e^{ij} L_i H_d E_j + \frac{1}{2}\kappa_{ij}(\tau) L_i H_u L_j H_u$$

In diagonal Y_e basis:

- Modular invariant holomorphic observables

$$I_{ij}(\tau) = \frac{\mathscr{M}_{ii}(\tau)\mathscr{M}_{jj}(\tau)}{(\mathscr{M}_{ij}(\tau))^2} = \frac{\kappa_{ii}\,\kappa_{jj}}{\kappa_{ij}^2} = \frac{m_{ii}(\tau,\bar{\tau})m_{jj}(\tau,\bar{\tau})}{(m_{ij}(\tau,\bar{\tau}))^2}$$

I_{ij} invariant under renormalization group

Chang, Kuo (2002)

Invariants in Toy Modular A_4 Model

MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

- Mass matrix in canonical basis:

$$m_\nu(\tau,\bar\tau) = (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)\frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_2(\tau) & -Y_3(\tau) \\ -Y_2(\tau) & 2Y_3(\tau) & -Y_1(\tau) \\ -Y_3(\tau) & -Y_1(\tau) & 2Y_2(\tau) \end{pmatrix} =: (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)v_u^2\begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{pmatrix}$$

- Invariants

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\big(Y_2(\tau)\big)^2}, \qquad I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\big(Y_3(\tau)\big)^2}, \qquad I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\big(Y_1(\tau)\big)^2}$$

- Algebraic constraint $\quad Y_2^2 + 2Y_1Y_3 = 0$

- Thus

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) = -2\big(1 + \frac{1}{3}j_3(\tau)\big)^3, \qquad I_{23}(\tau) = -\frac{32}{I_{23}}$$

Invariants in Toy Modular A_4 Model

- Two interesting relations: **RG invariant, independent of τ**

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) I_{23}(\tau) = -32$$

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

⇒ **sum rules among physical observables:**

RG invariant, τ independent

MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A$_4$ Model

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

$$I_{ij} = -2$$

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12} + c_{12}s_{13}s_{23}\right)^2 + \widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23} - s_{12}s_{13}s_{23}\right)^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12} + c_{12}s_{13}s_{23}\right) + \widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23} - \mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}\right) - \mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}s_{23}\right]^2}$$

$$\widetilde{m}_1 := m_1\,\mathrm{e}^{\mathrm{i}\,\varphi_1}$$

$$\widetilde{m}_2 := m_2\,\mathrm{e}^{\mathrm{i}\,\varphi_2}.$$

$$a_0 := \left(\widetilde{m}_1c_{12}^2 + \widetilde{m}_2s_{12}^2\right)c_{13}^2 + \mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}^2\;,$$

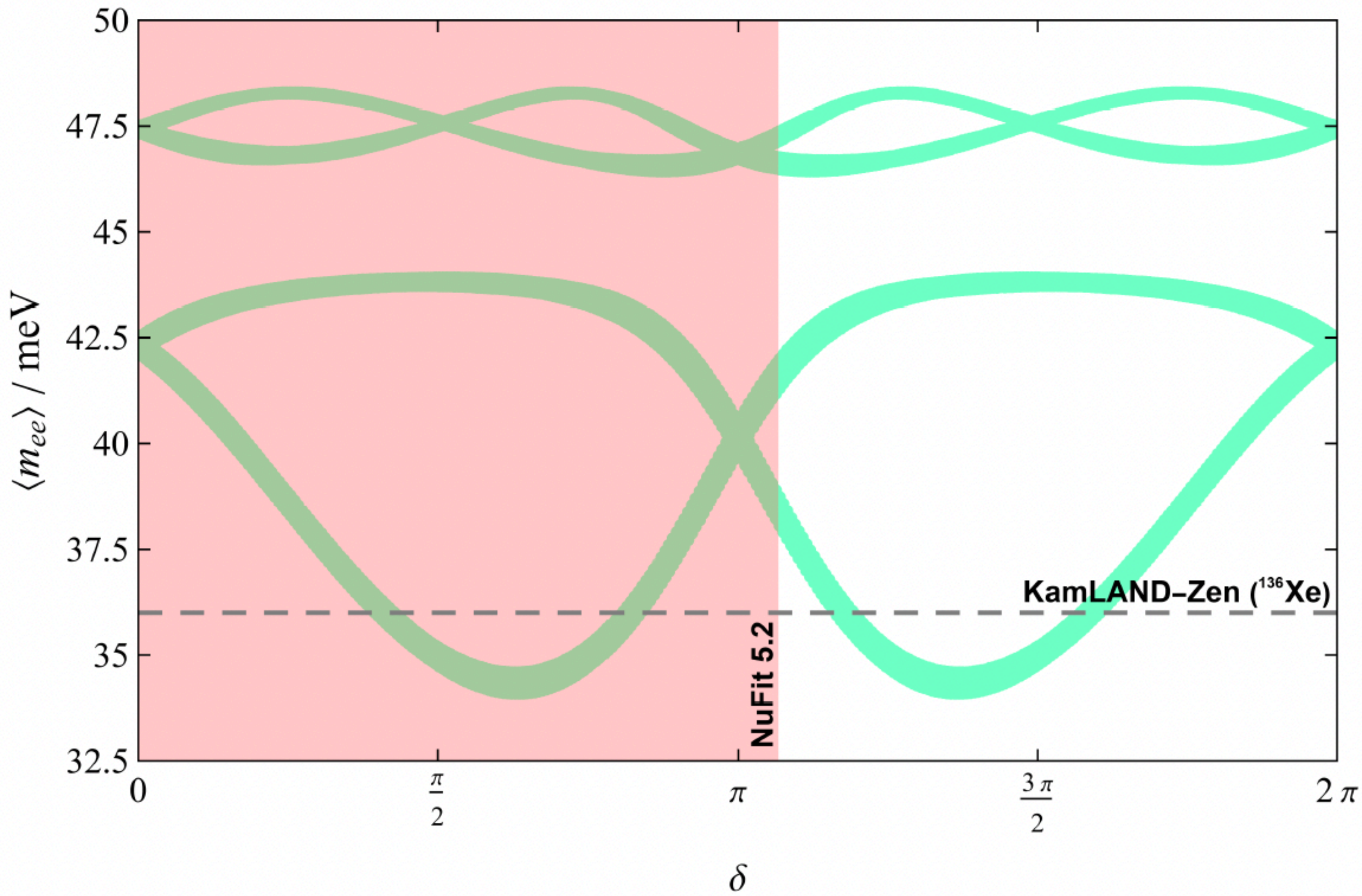
$$a_1 := \left[\left(\mathrm{e}^{2\,\mathrm{i}\,\delta}s_{12}^2 - c_{12}^2s_{13}^2\right)\sin(2\theta_{23}) - \mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right]\;,$$

$$a_2 := \left[\mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13} + \left(\mathrm{e}^{2\,\mathrm{i}\,\delta}c_{12}^2 - s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right]\;.$$

MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A_4 Model

- Predictions from $I_{12} = -2$ invariant, inverted Ordering



MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

Reality of $I_{12} \Rightarrow$

Constraints on CP phases:

Given $\delta \Rightarrow$

Majorana phases α_{12}, α_{13}

Invariants in Toy Modular A_4 Model

- No simultaneous solution for I_{ij} that is consistent with data
  - Agree with previous analysis by scanning parameter space (i.e. toy modular A4 model does not fit all data)
  - Here, arrived at conclusion without the need to scan

MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A_5 Model

MCC, X.-G. Liu, X.-Q. Li, O. Medina, M. Ratz (2024)

- In a model based on modular A5:

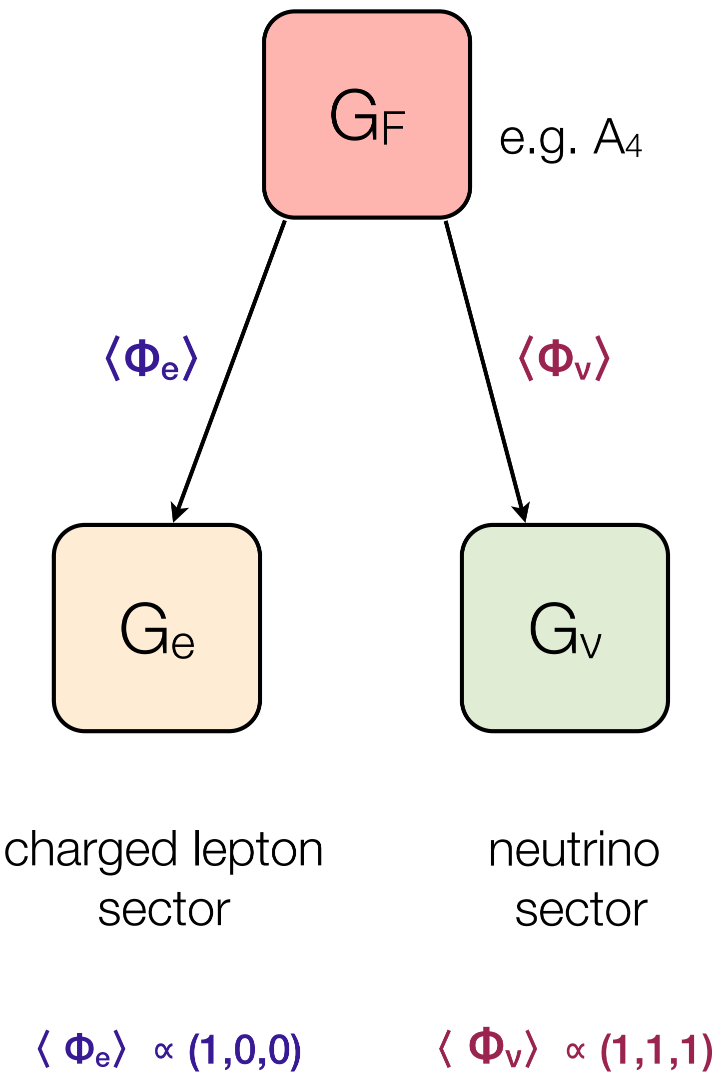
$$I_{12} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_4(\tau)}{Y_5^2(\tau)}\,, \qquad I_{13} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_3(\tau)}{Y_2^2(\tau)}\,, \qquad I_{23} = 6\frac{Y_3(\tau)Y_4(\tau)}{Y_1^2(\tau)}$$

- Algebraic relations among the invariants

$$\begin{aligned} 0 &= 4 + 18I_{12} + 18I_{13} + 9I_{12}I_{13} + I_{12}I_{13}I_{23}\,, \\ 0 &= 8 + 12I_{12} - 108I_{12}^2 + 12I_{13} + 414I_{12}I_{13} + 108I_{12}^2I_{13} - 108I_{13}^2 + 108I_{12}I_{13}^2 + 81I_{12}^2I_{13}^2 \\ &\quad - I_{12}^2I_{23} - I_{13}^2I_{23}\,. \end{aligned}$$

- Exchange symmetry: $I_{12} \leftrightarrow I_{13} \;\Rightarrow\; \mu - \tau$ symmetry built in

Flavor Model Structure



- interplay between the symmetry breaking patterns in two sectors lead to lepton mixing (BM, TBM, ...)
- symmetry breaking achieved through flavon VEVs
- each sector preserves different residual symmetry
- full Lagrangian does not have these residual symmetries

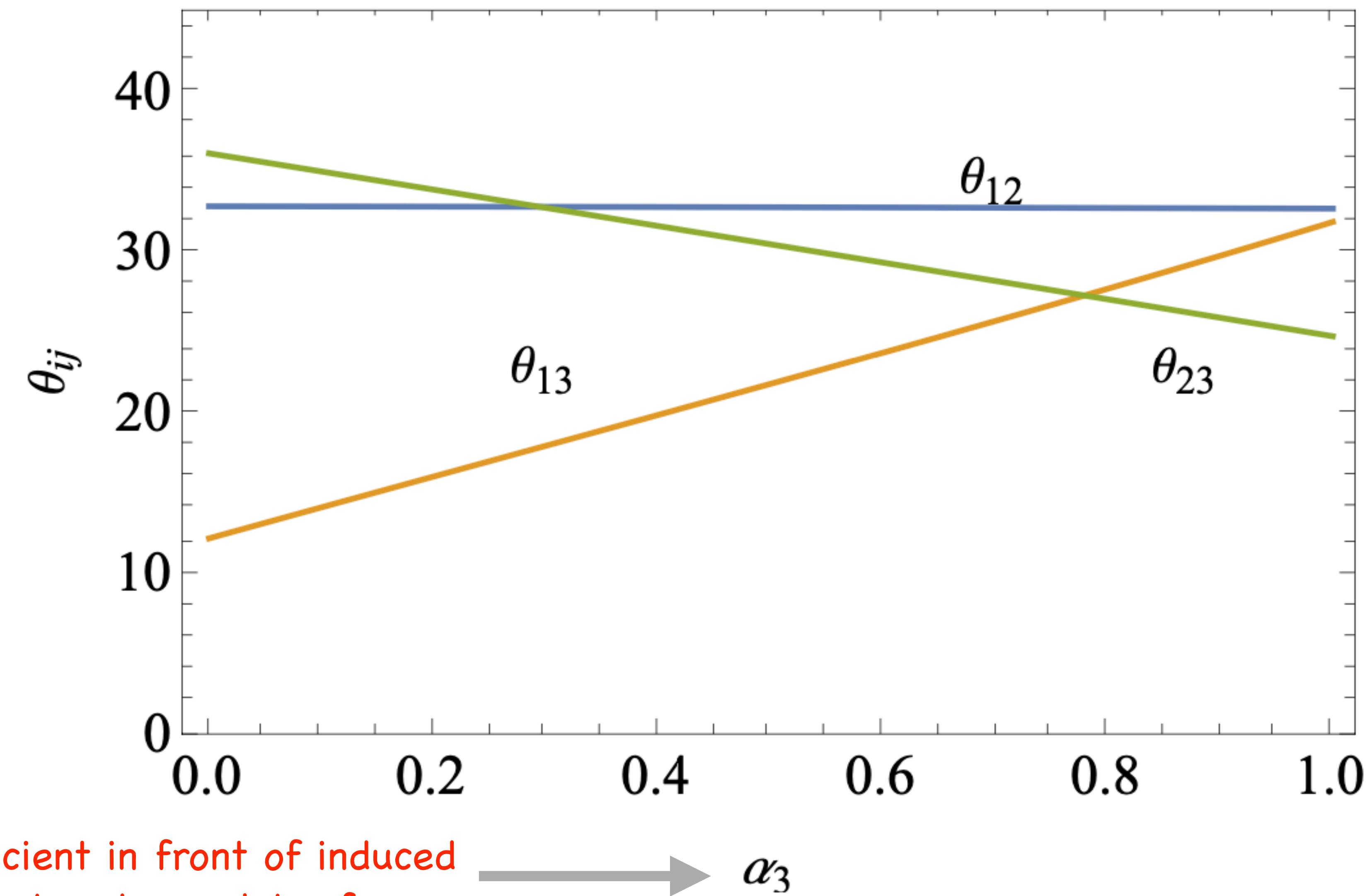
$\Rightarrow$ Corrections to model predictions

Corrections to Kinetic Terms

- Corrections to the kinetic terms induced by family symmetry breaking generically are present, should be properly included Leurer, Nir, Seiberg (1993); Dudas, Pokorski, Savoy (1995); Dreiner, Thomeier (2003)
  - cannot be prevented by conventional symmetries
  - could be sizable for neutrino mass models based on traditional discrete family symmetries, e.g. A_4 M.-C.C, M. Fallbacher, M. Ratz, C. Staudt (2012)
    - nontrivial flavor structure can be induced
    - non-zero CP phase can be induced
    - Presence of additional undetermined parameters
- also present in models based on modular flavor symmetries, induced by modular form MCC, Ramos-Sánchez, Ratz (2019)
  - corrections could be sizable

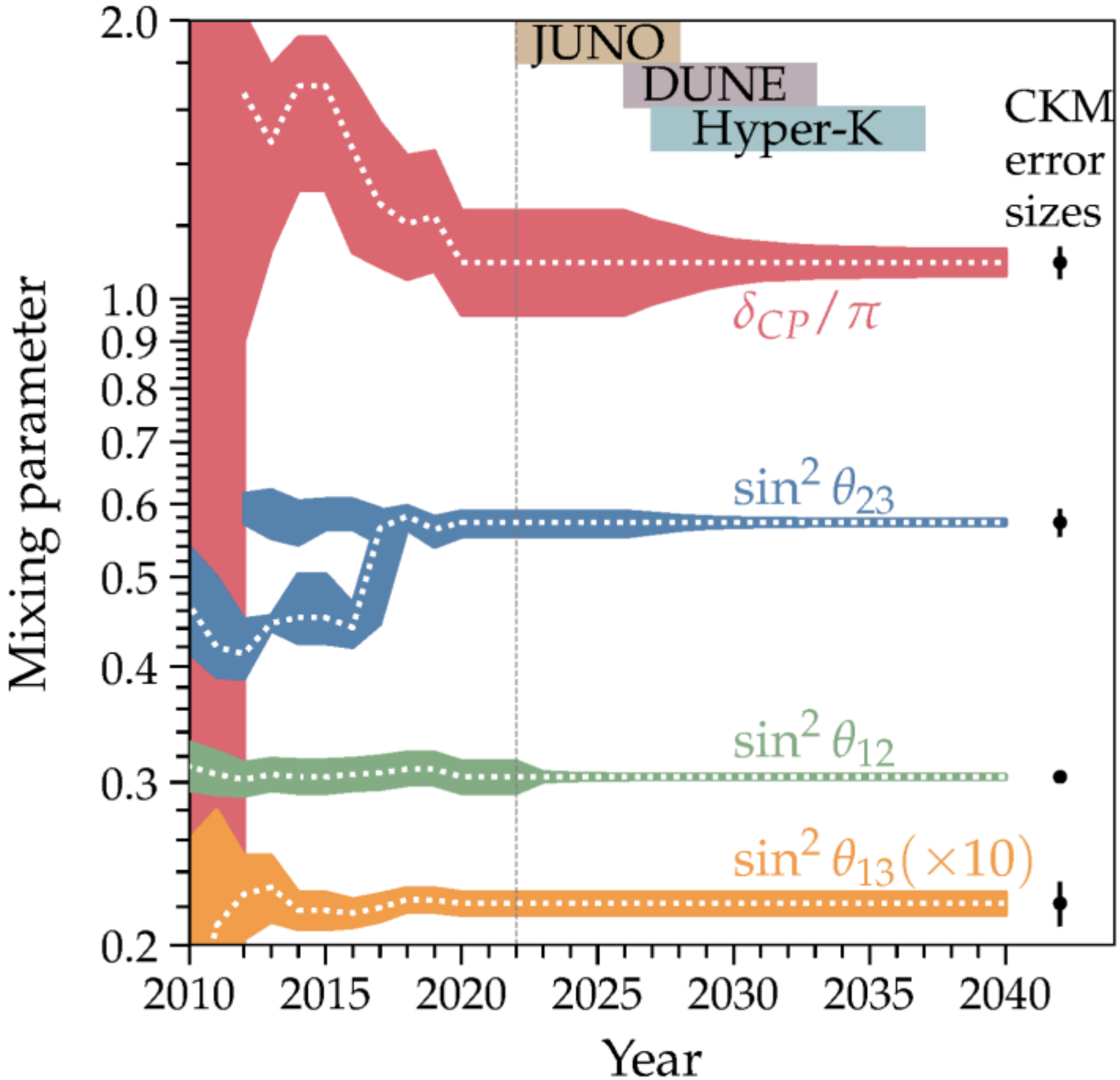
Kähler Corrections in Modular A4 Model

M.-C.C., Ramos-Sánchez, Ratz (2019)



Coefficient in front of induced operators by modular form

Experimental Precision



Are precision in model predictions compatible with experimental precision?

Figure from Song, Li, Argüelles, Bustamante, Vincent (2020)

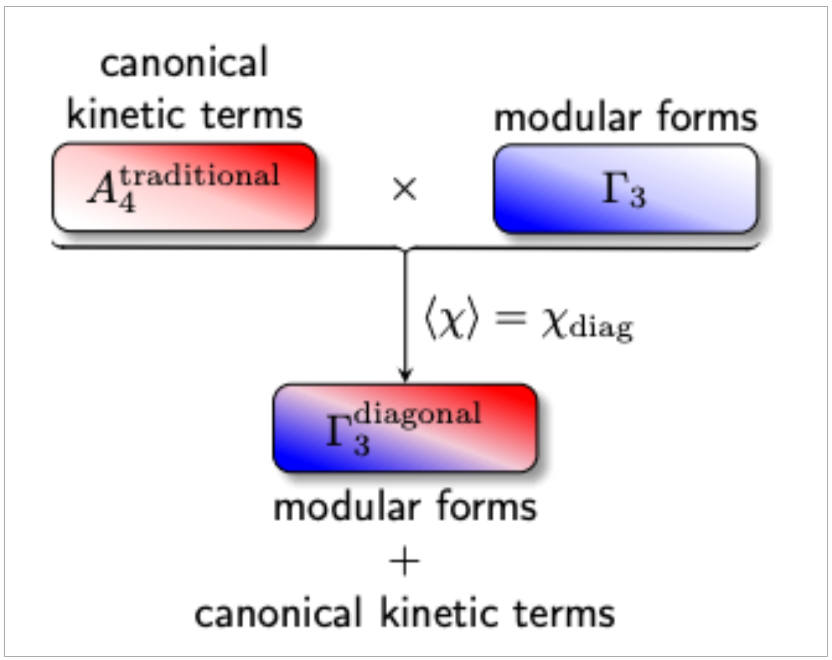
Quasi-Eclectic Modular Symmetry

MCC, Knapp-Pérez, Ramos-Hamud, Ramos-Sánchez, Ratz, Shukla (2021)

- Quasi-eclectic setup:

$$G_{\text{quasi-eclectic}} = G_{\text{traditional}} \times G_{\text{modular}} = A_4 \times \Gamma_3$$

- Symmetry Breaking



- After Symmetry Breaking
  - Corrections involving only Y: absent to all orders, due to traditional A4 symmetry (corrections in modular setup)
  - Corrections involving flavon VEV: highly uppressed (corrections in traditional setup)

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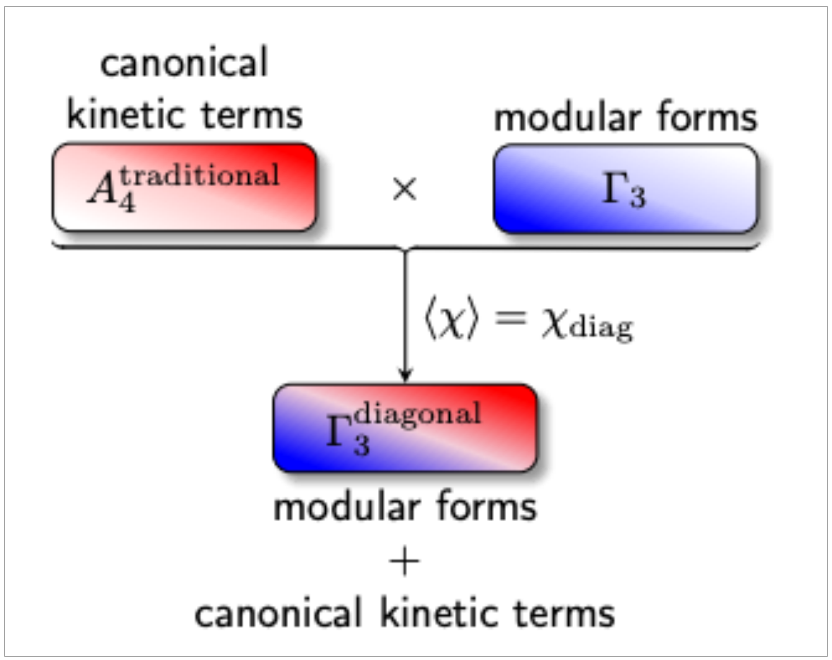
- Symmetry Breaking



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Corrections under control

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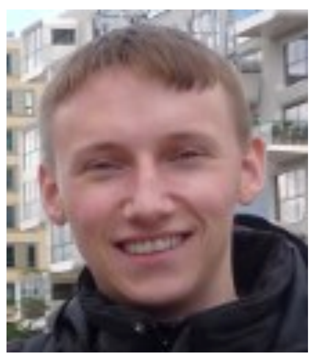
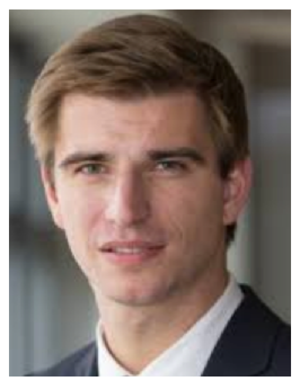


Maximilian Fallbacher
(former TUM Grad)



Christian Staudt
(former TUM Grad)

Saúl Ramos-Sánchez
(UNAM, Mexico)

Michael Ratz
(UCI)

Conclusions

- Fundamental origin of fermion mass & mixing patterns still unknown
- Uniqueness of Neutrino masses offers exciting opportunities to explore BSM Physics
- Modular Flavor Symmetries:
  - Significant reduction of the number of parameters
  - τ-independent RG Invariants: robust sum rules among physical observables, independent of renormalization scale, model parameters
  - In quasi-eclectic setup: corrections can be greatly reduced to the level compatible with experiment uncertainty
- Top-down connection:

  Talk by Michael Ratz on Friday

  - Modular flavor symmetries from strings e.g. Baur, Nilles, Trautner, Vaudrevange (2021)
  - Modular flavor symmetries from magnetized tori e.g. Almumin, MCC, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla (2021)
- Diversity drives intellectual excellence: ICISE engagement important

NEUTRINO 2026

University of California, Irvine, U.S.A.

About Irvine, California

a metropolitan city located at about 40 miles (64 km) south of Los Angeles, 70 miles (112 km) north of San Diego, on the beautiful coast of the Pacific Ocean with 11,000 ft (3500 m) towering San Bernadino Mountains in its backdrop.

70th Anniversary of Neutrino Discovery

by George Cowan and Fred Reines. Fred Reines (1995 Nobel Laureate) was the founding Dean of School of Physical Sciences at UC Irvine.

Contact Us

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