Talk
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SIDM as a solution to small scale crisis

Speaker: Utkarsh Patel(PhD Physics)
PhD Supervisor: Dr. Sudhanwa Patra

Indian Institute of Technology, Bhilai

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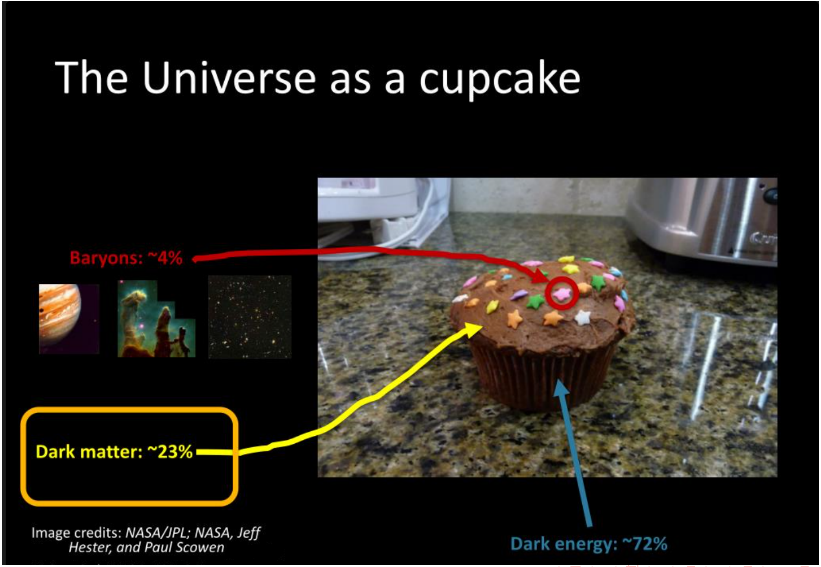
Outline

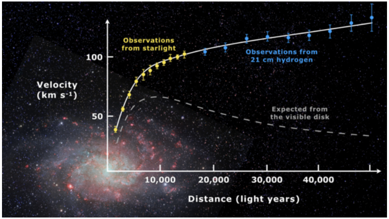
1. General review of DM
2. Freeze-out Mechanism and WIMP Miracle
3. Shortcomings at the small scale using WIMP scenario
4. Resolution using SIDM
5. Incorporating SIDM candidate in a model framework
6. Conclusion

~~Pie Chart~~ Cup-Cake Chart



First Detections

- Observations made by Vera Rubin
- Unusually high angular velocity
- Matter beyond the visible



The observations made by Rubin and fellow astronomers indicated towards the presence of non-baryonic matter in the galaxies. Similar observations were made for other galaxies as well. Dark Matter was hypothesised.

Model from Basic Physics

Velocity for galactic arms beyond a certain distance was found to be constant. Using basic Newtonian physics, it can be shown that matter density in a galaxy falls as inverse of radial distance squared.

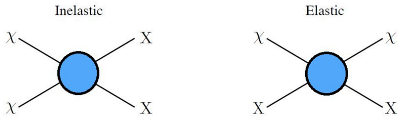
$$\boxed{v_r = \sqrt{\frac{GM}{r}} = \text{constant}}$$

In order to accomodate these results, the Mass parameter has to have a linear dependence on radius(r). Thus,

$$\boxed{\rho(r) \propto \frac{M(r)}{r^3} \sim \frac{1}{r^2}}$$

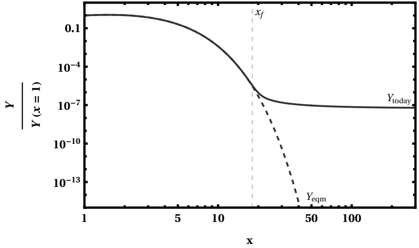
Possible scattering processes

- Inelastic leads to dark matter annihilations whereas elastic process is the interaction of DM with standard model particle, if any.
- Depending on the velocity of the DM particle, time scale for kinetic decoupling is decided
- Cold dark matter is better suited to explain structures than Hot Dark Matter



freeze-out mechanism

- Density of DM particles freeze-out in the universe
- Boltzmann equation gives a relation between present number density and the density during freeze-out
- Calculated results matches well at the weak-scale level with the observational results to give WIMP miracle



Note: Figure shows the dependence of DM number density Y on the scaled time parameter x. The plot has been generated using MATHEMATICA for the case of constant cross-section.

$$\boxed{\Omega_\chi = \frac{ms_{\text{today}} Y_{\text{today}}}{\rho_{cr}} \rightarrow \Omega_\chi h^2 \sim \frac{10^{-26} cm^3/s}{\langle \sigma v \rangle}}$$

$$\boxed{\Omega_\chi h^2 \simeq 0.1 \left(\frac{0.01}{\alpha}\right)^2 \left(\frac{m_\chi}{100\text{GeV}}\right)^2} \quad (1)$$

NFW regime

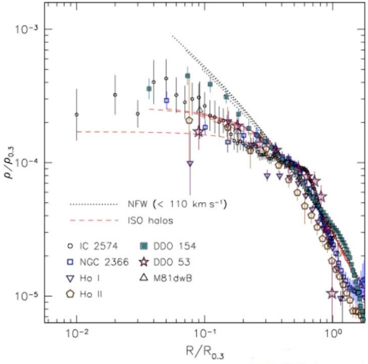
Navarro–Frenk–White used the freeze-out calculations to simulate a model for a galactic system with **WIMP Cold Dark Matter**. The relation they found out for the matter density vs. radial density is given as:

$$\boxed{\frac{\rho(r)}{\rho_{crit}} = \frac{\delta_c}{(r/r_s)(1+r/r_s)^2}}$$

Here, r_s is the scale radius, δ_c is a characteristic density. For radius values smaller than r_s, r/r_s would be smaller than 1, thus density(ρ) with radius would follow a r^{-1} dependence whereas when the radius is larger than r_s, then r/r_s would be larger than 1, hence the density with radius would follow a r^{-3} dependence. At, exactly $r = r_s$, the graph would have a r^{-2} relationship between density and radial distance, which was predicted using the basic physics by reflecting upon the observations made by rubin.

Core-Cusp Issue

Results from NFW profile simulation are shown via black dashed curves, and other observational results shown here suggest a constant inner core. This discrepancy is referred as **Cuspy-Core Problem**.

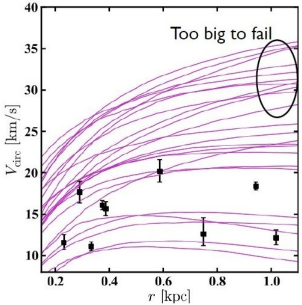


- Density profile: $\rho \sim r^{\alpha}$
  Predicted: $\alpha \sim -1$
  Observed: $\alpha = -0.29 \pm 0.07$
- Observed central density shows flattened cores.
- N-body simulations give out cuspy profiles.

Figure Courtesy of: Se-Heon Oh, W. J. G. de Blok, Elias Brinks, Fabian Walter and Robert C. Kennicutt — "Dark and luminous matter in THINGS dwarf galaxies"

Too-big-to-fail issue

$$V \sim \sqrt{\frac{GM_<}{r}}$$



Biggest predicted satellites from collisionless CDM simulations

Brightest observed satellites in the MW

Figure Courtesy of: Michael Boylan-Kolchin, James S. Bullock and Manoj Kaplinghat — "Too big to fail? The puzzling darkness of massive Milky Way subhaloes"

- Most massive subhalos in collisionless CDM simulations are well-enough dense to host observable galaxies in the Milky Way as it is easier for stars to form in massive subhalos.
- But all those massive DM halos are missing any stars or visible matter, which is highly unlikely.

Self-interacting Dark Matter

The challenges faced by CDM model can be overcome if Dark Matter is assumed to be self-interacting.

- CDM has a cooler core compared to the outer region.
- DM self-interactions cause heat transfer from the hot outer region to the cold inner region.
- Kinetic and thermal stability of the halo.
- A shallower density profile and more spherical halo shape.

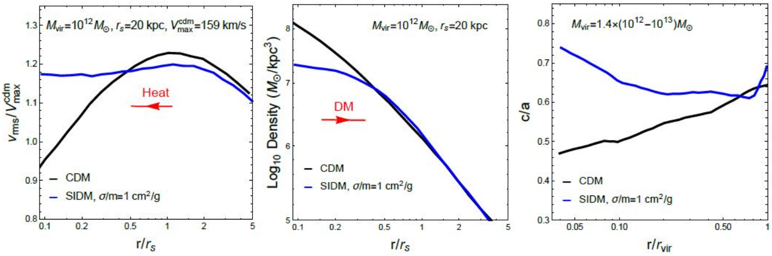


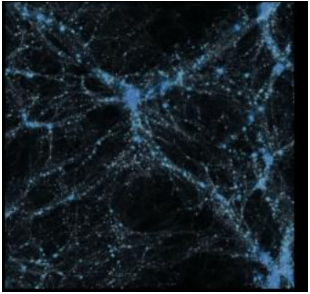
Image courtesy of: Sean Tulin and Hai-Bo Yu — "Dark Matter Self-interactions and Small Scale Structures"

Particle Physics for SIDM

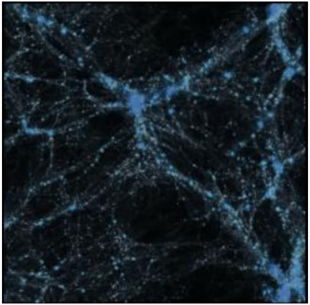
- The assumption of DM self-interactions force us to incorporate the DM elastic scattering rate and cross-section in the Boltzmann equation used for performing freeze-out calculations.
- Calculations done by the researchers show that a typical scattering cross-section (σ) needed to solve small scale crisis is much larger than the cross section for weak scale.
  - $\sigma_{EW} \sim 10^{-36}\text{cm}^2$
  - $\sigma_{\text{calculated}} \sim 1 \times 10^{-24}\text{cm}^2(m_\chi/GeV)$
- This hints towards a non-WIMP type DM candidate which atleast in the small scales have a large scattering cross section.

SIDM Introduction

- Elastic scattering– need cross-section $\sim 1cm^2/g(10^{12}$ times stronger than the weak force) to be interesting.[7]
- Original formulation (Spergel & Steinhardt 2000)[6] hard-sphere elastic scattering.
- In vogue now: on particle side(hidden-sector models, models with new gauge symmetries, Sommerfeld-enhanced dark matter)–generally velocity-dependent.



(a) 10 Mpc/h slice, CDM[5]



(b) 10 Mpc/h slice, σ/m=$1cm^2/g$[5]

Figure: At large scales, SIDM simulates same result as CDM for a cross-section of order $1cm^2/g$ at small scales.

Model with a light mediator as the interaction carrier

- The simplest model can induce self-interaction through additional weakly-coupled states of a charged DM under a spontaneously broken $U(1)$ symmetry.
- The DM stability here is handled automatically by the charge conservation.
- As a by-product, gauge boson exchange mediates self-interactions.

The interaction term in Lagrangian can be of following type:

$$\boxed{\mathscr{L}_{int} = g_\chi \bar{\chi}\gamma^\mu \chi \phi_\mu} \text{ vector mediator, } \boxed{\mathscr{L}_{int} = g_\chi \bar{\chi}\chi\phi} \text{ scalar mediator} \tag{2}$$

Here, χ is DM particle, ϕ is the mediator particle and g_χ is the coupling constant.

Calculating Transfer cross-section for DM interactions

For a non-relativistic DM scattering defined by a Yukawa potential:

$$V(r) = \pm \frac{\alpha_X}{r} e^{-m_\phi r} \tag{3}$$

- Here, scalar mediator type interactions[2] $\rightarrow$ purely attractive.
- Vector mediator type interaction[2] $\rightarrow$ can be either attractive or repulsive.

The transfer cross-section, computed perturbatively in α_X from equation[3], is given by[2]:

$$\sigma_T^{\text{Born}} = \frac{8\pi\alpha_X^2}{m_X^2 v^4} \left(log(1 + m_x^2 v^2 / m_\phi^2)) - \frac{m_X^2 v^2}{m_\phi^2 + m_X^2 v^2} \right) \tag{4}$$

This works for both attractive and repulsive potentials, where v is the relative velocity. This expression is only valid within the approximation, $\alpha_X m_X / m_\phi << 1$.

Transfer cross-section continued..

- Within the non-perturbative regime, analytic formulae like equation[4] are obtained within the classical limit $(m_X v/m_\phi >> 1)$[2, 3, 4], given by

$$\boxed{\sigma_T^{clas} = \begin{cases} \frac{4\pi}{m_\phi^2}\beta^2 ln(1+\beta^{-1}), & \beta \lesssim 10^{-1} \\ \frac{8\pi}{m_\phi^2}\beta^2/(1+1.5\beta^{1.65}), & 10^{-1} \lesssim \beta \lesssim 10^3 \\ \frac{\pi}{m_\phi^2}(ln\beta + 1 - \frac{1}{2}ln^{-1}\beta)^2, & \beta \gtrsim 10^3 \end{cases}}$$

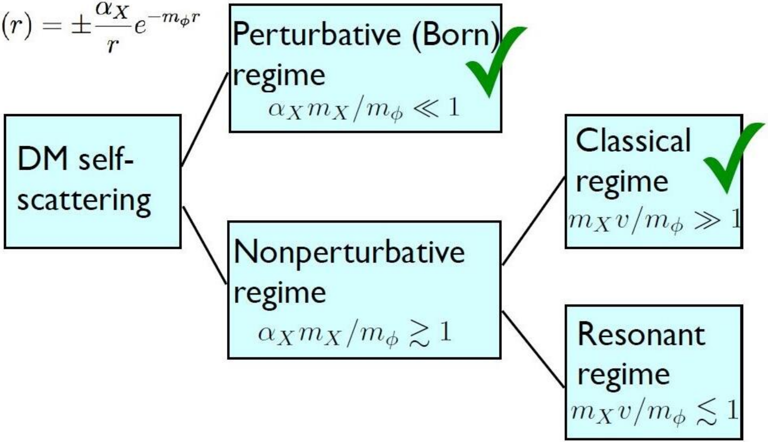
Here, $\beta \equiv 2\alpha_X m_\phi/(m_X v^2)$.

- For a large parametric range, neither the Born nor classical approximations are valid.
- governed by the conditions $\alpha_X m_X/m_\phi \gtrsim 1$ and $m_X v/m_\phi \lesssim 1$.
- This **"resonant regime"** is computed by solving the Schrödinger equation directly using a partial wave analysis.

Big Picture

For a Yukawa-type potential

$$V(r) = \pm \frac{\alpha_X}{r} e^{-m_\phi r}$$



Numerical Methodology for SIDM

Focus is:

1. To compute DM self-interaction cross-section in the resonant regime (σ)
   - In terms of differential cross-section, $\sigma = \int d\Omega(\frac{d\sigma}{d\Omega})$
   - A quantity called transfer cross-section is a better measure of particle interactions in the light mediator scenario. It is defined as:

$$\boxed{\sigma_T = \int d\Omega(1 - cos\theta)\frac{d\sigma}{d\Omega}} \quad (5)$$

   - $\frac{d\sigma}{d\Omega}$ is better understood in terms of partial wave analysis (detailed analysis in the next slide...).
   - The final form of expression involves a phase factor which requires us to solve the Schrodinger equation for a two body system with potential V(r).
2. Also to investigate velocity-dependence of DM Scattering

Expression for differential cross-section

- Quantum mechanics partial wave analysis

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_\ell}{dr}\right)+\left(k^2-\frac{\ell(\ell+1)}{r^2}-2\mu V(r)\right)R_\ell=0$$

- Transfer cross section

$$\frac{d\sigma}{d\Omega}=\frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}P_\ell(\cos\theta)\sin\delta_\ell\right|^2 \qquad \sigma_T=\int d\Omega\,(1-\cos\theta)\,\frac{d\sigma}{d\Omega}$$

$$\frac{\sigma_T k^2}{4\pi}=\sum_{\ell=0}^{\infty}\left[(2\ell+1)\sin^2\delta_\ell-2(\ell+1)\sin\delta_\ell\sin\delta_{\ell+1}\cos(\delta_{\ell+1}-\delta_\ell)\right]$$

$$\downarrow$$

$$\frac{\sigma_T k^2}{4\pi}=\sum_{\ell=0}^{\infty}(\ell+1)\sin^2(\delta_{\ell+1}-\delta_\ell)$$

Here, δ (the phase shift) is obtained by solving Schrodinger equation for the radial wavefunction $R_\ell(r)$.

Solving the Schrodinger Equation

1. The phase shift δ_ℓ parametrizes the asymptotic solution for $R_\ell(r)$, given by,

$$\lim_{r\to\infty} R_\ell(r) \propto cos\delta_\ell j_\ell(kr) - sin\delta_\ell n_\ell(kr)$$

2. Input
   - A potential type: Here, we have taken Yukawa Potential $V(r) = \pm\frac{\alpha_\chi}{r}e^{-m_\phi r}$.
   - Boundary Conditions.
3. The Schrödinger equation describing the evolution of the reduced two-particle wave function for particles of mass m_ψ and impulsion $\vec{k}$ in a potential V is given by,

$$\boxed{\frac{1}{m_\psi}\nabla^2\psi_k - V(r)\psi_k = -\frac{k^2}{m_\psi}\psi_k,}$$

Sommerfeld Effect

1. An important quantum phenomenon that can lead to enhanced dynamics in particle interactions.
2. Occurs due to the exchange of multiple mediator particles between the two interacting particles in a potential before the actual interaction.
3. Can enhance the scattering cross-section, if the interaction involves an attractive external potential.

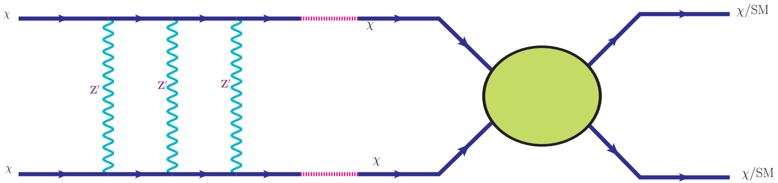
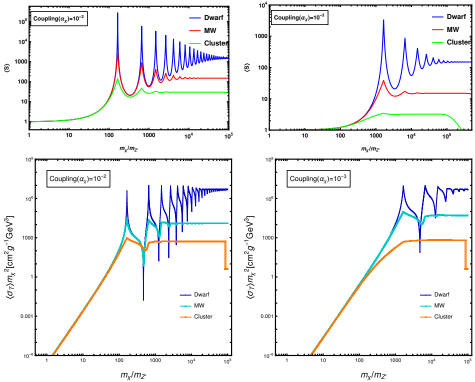


Figure: Sommerfeld effect in SIDM(χ) via a light mediator.



The expression for $\langle S \rangle$:

$$\langle S \rangle = \int \frac{d^3 v}{(2\pi v_0^2)^{3/2}} e^{-\frac{1}{2} v^2 / v_0^2} S_s$$

with S_s being the s-wave Sommerfeld enhancement factor and is expressed as:

$$S_s = \frac{\pi}{a} \frac{\sinh(2\pi ac)}{\cosh(2\pi ac) - \cos(2\pi \sqrt{c - (ac)^c})}$$

with $a = v/2\alpha_X$ and $c = 6\alpha_X m_X / \pi^2 m_\phi$

Figure shows the dependence of thermally-averaged sommerfeld factor on the ratio of DM to mediator mass in the upper half. We see a strong resonance features for a ratio of around 100-10000 in both the $\alpha = 10^{-2}$ and 10^{-3} cases. For a comparison, the dependence of thermally-averaged scattering cross-section on the same ratio has been plotted in the lower half of the image. Thus, we can see a clear co-relation between the enhancements in the thermally averaged DM scattering cross-section and DM annihilation cross-section.

Calculations using Numerical Approach

- Upon defining certain variables and functions for convenience, the final form of Schrodinger equation looks like:

$$\left(\frac{d^2}{dx^2} + a^2 - \frac{l(l+1)}{x^2} \pm \frac{1}{x} e^{-x/b}\right) \chi_l(x) = 0.$$

Here, $\chi_l \equiv rR_{k,l}, \quad x \equiv \alpha_{\rm med} m_\psi r, \quad a \equiv \frac{v}{2\alpha_{\rm med}}, \quad b \equiv \frac{\alpha_{\rm med} m_\psi}{m_{\rm med}}$

- Thus, in the next slide let me provide a comparison between the 3 regimes for a particular velocity value $10 km/s$, which corresponds to the DM velocity at the small scale.
- From the figures,[1a] and [1b] and reference[7], it can be seen that σ/m_X around $1cm^2/g$ at small scale can not only resolve small scale issues but also leaves the large scale structure undisturbed if the cross-section fades away with DM velocity.

Comparison between the 3 approaches

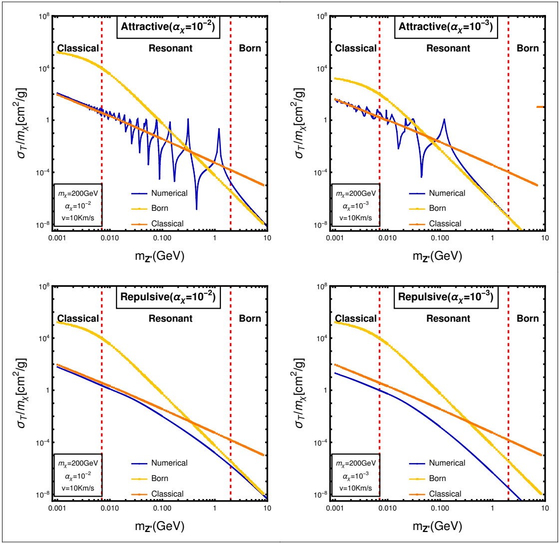


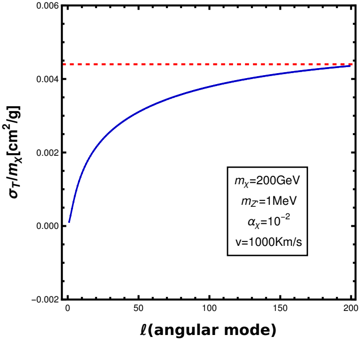
Figure: The numerical results(blue) shows convergences to classical(green) and born(orange) lines in the Born and Classical regions. It can also be seen that the resonant and anti-resonant peak pattern of numerical results due to the formation of bound states(for an attractive potential) is not produced in the analytical method and also not in the case of repulsive type potential. These plots have been generated using **MATHEMATICA** by solving 2-body interaction problem numerically.

Unit Conversion:

$1GeV^{-2} = 4x10^{-28}cm^2$

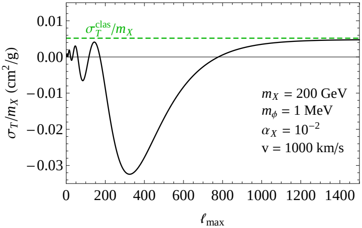
$1GeV = 1.8x10^{-24}g$

Improvement over the previous results



(a) Numerical calculation of $\sigma_T/m_{Z'}$(blue curve) depicting a saturation to classical analytical result(dotted red line) with increasing ℓ modes for our work. The formula for transfer cross-section used here is given as[1]:

$$\frac{\sigma_T k^2}{4\pi} = \sum_{\ell=0}^{\infty}(\ell+1)\sin^2(\delta_{\ell+1}-\delta_\ell)$$



(b) Numerical calculation of $\sigma_T/m_{Z'}$(blue curve) depicting a saturation to classical analytical result(dotted red line) with increasing ℓ modes for the work in reference[7]. The formula for transfer cross-section used there is given as:

$$\frac{\sigma_T k^2}{4\pi} = \sum_{\ell=0}^{\infty}\big[(2\ell+1)\sin^2\delta_\ell \\ -2(\ell+1)\sin\delta_\ell\sin\delta_{\ell+1}\cos(\delta_{\ell+1}-\delta_\ell)\big)$$

Numerical Methodology for SIDM

Focus is:

1. To compute DM self-interaction cross-section in the resonant regime (σ)
2. To investigate velocity-dependence of DM Scattering
   - The most vital element of any SIDM framework is the dependence of cross-section(σ_T) on Dark Matter velocity.
   - The Dark Matter particles have varied average velocities across all the scales of the universe.
   - This relationship is a increasing function where the particle velocity increases from the sub-galactic to dwarf scale and from dwarf to galactic and cluster scale.
   - In the next slide, let us see the graphs showing the dependence of DM cross-section on DM velocity using the Classical and Born formulae.

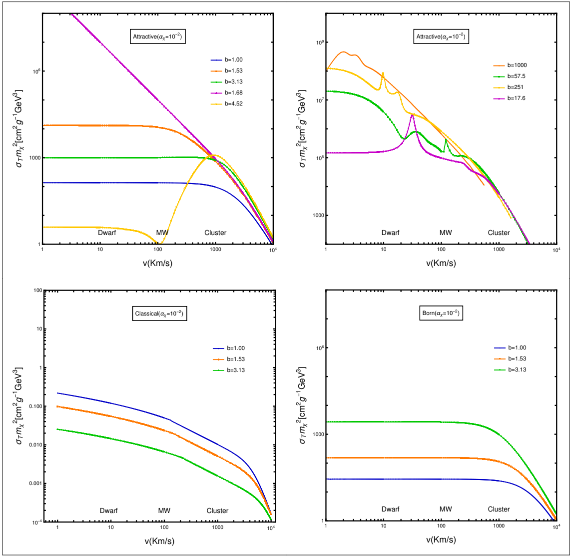
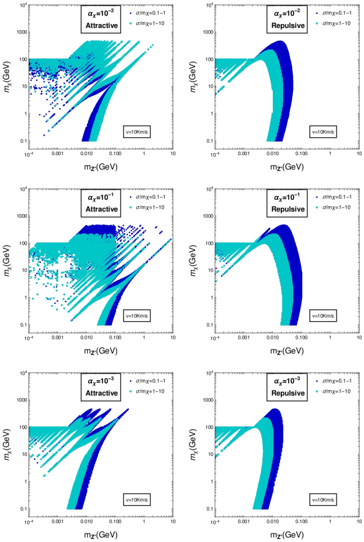


Figure: In these figures, the parameter 'b' is defined as $\alpha_X m_X/m_\phi$. For a fixed value of α, the ratio $\frac{m_X}{m_\phi}$ is also fixed. It is evident from the graphs that for all the different values of parameter 'b', the transfer cross-section σ_T is strongly suppressed for large values of velocity(i.e. at the large scales of the universe). Thus, at larger velocities, the transfer cross-section becomes independent of m_X/m_ϕ and $\sigma_T m_X^2 \propto v^{-4}$. This suppressed value of cross-section is desirable and will lead to the usual WIMP(Weakly Interacting Massive Particle) scenario at the large scales.

Parameter space for the framework



- The plots has been made using Hulthén potential, which is a proxy for original Yukawa-type potential.
- The Hulthén potential behaves similar to Yukawa and provides an solvable analytical formula for calculating transfer cross-section in the $\ell = 0$ limit case.
- The chosen cross-section(σ_T/m_χ) values are between 0.1 10, which have been found to solve small scale crisis and as the behaviour of cross-section changes at large scales(given its velocity dependence), the allowed parameter space works for all the scales.
- Due to the Sommerfeld enhancement, the resonant peak and anti-resonant valley nature is distinctive visible for the attractive-type potential case.

Incorporating SIDM in a theoretical framework

Gauge fields	$SU(2)_L$ $\vec{W}^\mu$	$U(1)_Y$ B^μ	$U(1)_{\mathbb{L}}$ $(Z_{\mathbb{L}}^\mu)$
Fermions (Other than SM)			
ξ_L	**1**	0	$-4/3$
η_L	**1**	0	$-1/3$
$\zeta_{kR}(k=1,2)$	**1**	0	$2/3$
$\Psi_L = \begin{pmatrix}\Psi_L^+ \\ \Psi_L^0\end{pmatrix}$	**2**	$1/2$	$3/2$
$\Psi_R = \begin{pmatrix}\Psi_R^+ \\ \Psi_R^0\end{pmatrix}$	**2**	$1/2$	$3/2$
$\Sigma_L = \frac{1}{\sqrt{2}}\begin{pmatrix}\Sigma_L^0 & \sqrt{2}\Sigma_L^+ \\ \sqrt{2}\Sigma_L^- & -\Sigma_L^0\end{pmatrix}$	**3**	0	$-3/2$
ρ_L	**1**	0	$-3/2$
Scalar fields			
H	**2**	$1/2$	0
ϕ_1	**1**	0	1
ϕ_2	**1**	0	2

Table: Gauge theory of leptons with field content and their transformations under the gauge group $SU(2)_L \times U(1)_Y \times U(1)_L$. These fields are chiral in nature and are color singlets while the hypercharge is given by the relation, $Q = T^3 + Y$.

- The scalar fields ϕ_1 and ϕ_2 give masses to new fermions after the spontaneous breaking of $U(1)_{\mathbb{L}}$ symmetry.
- The fermion mass matrix in this framework has a Dirac form. Hence, the model contains two Dirac mass eigenstates, denoted by χ_1 and χ_2, the lightest of which(χ_1) could be a **Dark Matter candidate**.
- The vector($g_{\chi V}$) and axial($g_{\chi A}$) couplings of the dark matter particle to Z' constitutes its self-interaction.
- Here, the interaction due to vector coupling ($g_{\chi V}$), given in the form $g_{\chi V}\bar{\chi}\gamma^\mu\chi Z'$ is of great importance to our framework.

Conclusions

- SIDM stands as a good alternative to WIMP as it is highly plausible for a theory to have particles that can self-interact given the disagreement of non self-interacting DM at small scales.
- The detailed numerical study of self-interacting scattering cross-section under a potential shows a rich dynamics with respect to mediator mass.
- The numerical analysis of SIDM scenario under a Yukawa potential is taken as one of my PhD project work and the complete work can be found at **arxiv:2204.11551**
- As an extension to this project, we have done a detailed study of SIDM in a leptophilic extension of the Standard Model along with the detection constraints involved and would be out in few weeks.

[1] Matthew R. Buckley and Patrick J. Fox. Dark Matter Self-Interactions and Light Force Carriers. *Phys. Rev. D*, 81:083522, 2010.

[2] Jonathan L. Feng, Manoj Kaplinghat, and Hai-Bo Yu. Halo-shape and relic-density exclusions of sommerfeld-enhanced dark matter explanations of cosmic ray excesses. *Physical Review Letters*, 104(15), Apr 2010.

[3] S. A. Khrapak, A. V. Ivlev, G. E. Morfill, and S. K. Zhdanov. Scattering in the attractive yukawa potential in the limit of strong interaction. *Phys. Rev. Lett.*, 90:225002, Jun 2003.

[4] Sergey A. Khrapak, Alexey V. Ivlev, and Gregor E. Morfill. Momentum transfer in complex plasmas. *Phys. Rev. E*, 70:056405, Nov 2004.

[5] Annika H. G. Peter, Miguel Rocha, James S. Bullock, and Manoj Kaplinghat. Cosmological simulations with self-interacting dark matter – II. Halo shapes versus observations. *Monthly Notices of the Royal Astronomical Society*, 430(1):105–120, 01 2013.

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[7] Sean Tulin, Hai-Bo Yu, and Kathryn M. Zurek. Beyond collisionless dark matter: Particle physics dynamics for dark matter halo structure. *Physical Review D*, 87(11), Jun 2013.