One-loop Feynman integrals with complex internal-masses, general space-time dimension and higher-power ε-expansion

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Outline

1. **Introduction**
2. **The calculation**
   $\rightarrow$ **One-loop two-point functions (in detail),**
   $\rightarrow$ **One-loop three-point functions (in detail),**
   $\rightarrow$ **One-loop four-point functions (present result),**
3. **Numerical results**
4. **Conclusions**

Based on: **K. H. Phan et al [**PTEP 2017 (2017) no.6, 063B06**,** arXiv:1710.11358v3 [hep-ph]**]**.

1. Future programs at the HL-LHC and the ILC [1] focus:
   - on studying the Brout-Englert-Higgs boson properties, top quark properties, vector boson productions,
   - searching for new physics signals.

   → **The mesurements are performed at high precision.**
2. Theoretical predictions including higher-order corrections to multi-particle processes at the colliders are required.
3. **Detailed evaluations of one-loop multi-leg and higher-loop are necessary.**

[1] Physics at a High-Luminosity LHC with ATLAS: arXiv:1307.7292; CMS: arXiv:1307.7135; ILC Technical Design Report: arXiv:1306.6352

Scalar one-loop integrals with complex internal masses?

Evaluating for Feynman diagrams involve internal unstable particles that can be on-shell

$\rightarrow$ we have to redefine their propagators with a complex mass term in the denominator,

$\rightarrow$ or perform **the perturbative renormalization** in the Complex-Mass Scheme **Denner et al, Nucl. Phys. B 724 (2005) 247**

Higher-power ε-expansion, in general space-time dimension?

$\rightarrow$ building block for evaluating two-loop (higher-loop) corrections.

$\rightarrow$ gain a good numerical stability (or cure small inversed Gram determinant problem) at one-loop corrections $\leftarrow$ **one-loop integrals with $d = 4(6, 8, \cdots) - 2\varepsilon$ must be taken into account:** **Davydychev, Phys. Lett. B263 (1991) 107-111, J. Fleischer et al, Phys. Rev. D 83 (2011) 073004, etc.**

References?

→ 't Hooft and M. Veltman,Nuclear Physics B **153** (1979) 365-401), A. Denner et al, Nucl. Phys. B**367** (1991) 637-656, **844** (2011) 199, D. T. Nhung et al, Comput. Phys. Commun. **180** (2009) 2258, etc.
→ Davydychev et al: Mellin-Barnes representaion J. Math. Phys. **33** (1992) 358-369 , Geometrical approaches J. Math. Phys. **39** (1998) 4299 .
→ Automatic Mellin-Barnes represent for multi-loop by J. Gluza et al, Comput. Phys. Commun. **177** (2007) 879, etc).
→ Tarasov et al: Recurrence relation for multi-loop integral in shifted dimension Nucl. Phys. Proc. Suppl. **89** (2000) 237, Nucl. Phys. B **672** (2003) 303
→ ⋯

→ The above calculations have been not completed to treat scalar one-loop integrals at general space-time dimension, higher-power ε-expansion, as well as complex internal masses, etc.

Based on the method [Kreimer et al, Z. Phys. C **54** (1992) 667, Int. J. Mod. Phys. A **8** (1993) 1797]

→ The detailed calculation for scalar one-loop integrals with complex internal masses are presented.

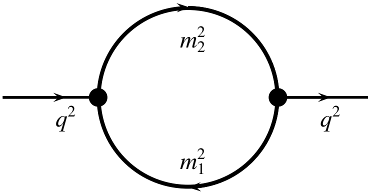
→ The analytic results for two-, and three- point functions are presented in terms of Carlson's function as well as the generalized hypergeometric (general space-time dimension D).

→ The analytic results for four-point functions are presented in terms of logarithm and dilogarithm functions ($D = 4 - 2\varepsilon$ at ε^0-expansion).

→ A package for numerical evaluations for one-loop written in `MATHEMATICA` and `FORTRAN` are built.

→ Numerical checks with `LoopTools` and `AMBRE/MB` are discussed.

One-loop two-point diagrams



$q = q(q_{10}, \overrightarrow{0}_{D-1})$,

[Kreimer et al, Z. Phys. C **54** (1992) 667, Int. J. Mod. Phys. A **8** (1993) 1797]

$$
\begin{aligned}
J_2 &= \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_0^{\infty} dl_\perp \frac{l_\perp^{D-2}}{\mathcal{P}_1 \mathcal{P}_2}, \\
\mathcal{P}_1 &= (l_0 + q_{10})^2 - l_\perp^2 - m_1^2 + i\rho, \\
\mathcal{P}_2 &= l_0^2 - l_\perp^2 - m_2^2 + i\rho.
\end{aligned}
$$

The internal masses are given

$$m_k^2 = m_{0k}^2 - i m_{0k}\, \Gamma_k,$$

with $k = 1, 2$. Γ_k are decay width of unstable particles.

Partitioning the integrand is as follows

$$\frac{1}{\mathcal{P}_1\mathcal{P}_2} = \frac{1}{\mathcal{P}_1(\mathcal{P}_2 - \mathcal{P}_1)} + \frac{1}{\mathcal{P}_2(\mathcal{P}_1 - \mathcal{P}_2)}.$$

We then make a shift $l_0' = l_0 + q_{10}$. The result reads

$$J_2 = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \frac{1}{2q_{10}} \int_{-\infty}^{\infty} dl_0 \int_0^{\infty} dl_\perp \left\{ \frac{l_\perp^{D-2}}{[l_0^2 - l_\perp^2 - m_2^2 + i\rho]\left[l_0 + \left(\frac{q_{10}}{2} - M_d\right)\right]} - \frac{l_\perp^{D-2}}{[l_0^2 - l_\perp^2 - m_1^2 + i\rho]\left[l_0 - \left(\frac{q_{10}}{2} + M_d\right)\right]} \right\},$$

with $M_d = \dfrac{m_1^2 - m_2^2}{2q_{10}}$.

After taking $l_\perp$-integration, we arrive at

$$\frac{J_2}{\Gamma\left(\frac{3-D}{2}\right)} = -\frac{\pi^{\frac{D-1}{2}}\, e^{i\pi(3-D)/2}}{2q_{10}} \int_{-\infty}^{\infty} dl_0 \left\{ \frac{(l_0^2 - m_2^2 + i\rho)^{\frac{D-3}{2}}}{l_0 + \left(\frac{q_{10}}{2} - M_d\right)} - \frac{(l_0^2 - m_1^2 + i\rho)^{\frac{D-3}{2}}}{l_0 - \left(\frac{q_{10}}{2} + M_d\right)} \right\},$$

with $M_d = \dfrac{m_1^2 - m_2^2}{2q_{10}}$.

For case of $m_1, m_2 \in \mathbb{R}$, or $\Gamma_1 = \Gamma_2$, the integrand may have singularity poles in real axis. We check that

$$(l_0^2 - m_2^2)|_{l_0 = -\left(\frac{q_{10}}{2} - M_d\right)^2} = \left(\frac{q_{10}}{2} - M_d\right)^2 - m_2^2 = \frac{\lambda(q_{10}^2, m_1^2, m_2^2)}{4q_{10}^2},$$

$$(l_0^2 - m_1^2)|_{l_0 = +\left(\frac{q_{10}}{2} + M_d\right)^2} = \left(\frac{q_{10}}{2} + M_d\right)^2 - m_1^2 = \frac{\lambda(q_{10}^2, m_1^2, m_2^2)}{4q_{10}^2},$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz, \quad \text{is the Källen function.}$$

$\Rightarrow$ Hadamard's finite part of these integrals will be cancelled each other.

The $\mathcal{R}$-function is defined as **[B. C. Carlson,** Special Functions of Applied Mathematics. **New York : Academic Press,1977]**

$$\int_r^\infty (x-r)^{\alpha-1} \prod_{i=1}^{k} (z_i + w_i x)^{-b_i} dx =$$

$$= \mathcal{B}(\beta-\alpha,\alpha)\mathcal{R}_{\alpha-\beta}\left(b_1,\cdots,b_k, r+\frac{z_1}{w_1},\cdots, r+\frac{z_k}{w_k}\right) \prod_{i=1}^{k} w_i^{-b_i}, \ \beta = \sum_{i=1}^{k} b_i.$$

Analytic formula for J_2 is presented

$$\frac{J_2}{\Gamma\left(3-\frac{D}{2}\right)} = \frac{\pi^{(D-1)/2} e^{i\pi(3-D)/2}}{2} \times \mathcal{B}\left(\frac{4-D}{2},\frac{1}{2}\right) \times$$
$$\times \left\{ \left(\frac{q^2+m_1^2-m_2^2}{2q^2}\right) \mathcal{R}_{\frac{D-4}{2}}\left(\frac{3-D}{2}, 1; -m_1^2 + i\rho, -\frac{(q^2+m_1^2-m_2^2)^2}{4q^2}\right)\right.$$
$$\left. + \left(\frac{q^2-m_1^2+m_2^2}{2q^2}\right) \mathcal{R}_{\frac{D-4}{2}}\left(\frac{3-D}{2}, 1; -m_2^2 + i\rho, -\frac{(q^2-m_1^2+m_2^2)^2}{4q^2}\right)\right\},$$

Euler's transformation **[B. C. Carlson,** Special Functions of Applied Mathematics. **New York : Academic Press,1977]**

$$\mathcal{R}_t(b_1, b_2, \cdots, b_N, z) = \prod_{i=1}^{k} z_i^{-b_i} \mathcal{R}_{-\beta-t}(b_1, \cdots, b_i + e_i, \cdots, b_N, z^{-1}),$$

$$\frac{J_2}{\Gamma\left(3-\frac{D}{2}\right)} = -\pi^{(D-1)/2} e^{i\pi(3-D)/2} \times \mathcal{B}\left(\frac{4-D}{2}, \frac{1}{2}\right) \times$$
$$\times \left\{ \frac{(-m_1^2+i\rho)^{\frac{D-3}{2}}}{q^2+m_1^2-m_2^2} \mathcal{R}_{-\frac{1}{2}}\left(\frac{5-D}{2}, 2; \frac{-1}{m_1^2-i\rho}, \frac{-4q^2}{(q^2+m_1^2-m_2^2)^2}\right) + \frac{(-m_2^2+i\rho)^{\frac{D-3}{2}}}{q^2-m_1^2+m_2^2} \mathcal{R}_{-\frac{1}{2}}\left(\frac{5-D}{2}, 2; \frac{-1}{m_2^2-i\rho}, \frac{-4q^2}{(q^2-m_1^2+m_2^2)^2}\right) \right\}$$

At threshold $q^2 = (m_1+m_2)^2$ or pseudo threshold $q^2 = (m_1-m_2)^2$:

$$\frac{J_2}{\Gamma\left(2-\frac{D}{2}\right)} = \pi^{(D-1)/2} e^{i\pi(3-D)/2} \left\{ \left(\frac{q^2+m_1^2-m_2^2}{4q^2}\right)(-m_1^2+i\rho)^{\frac{D-4}{2}} + (1 \leftrightarrow 2) \right\}.$$

This shows that J_2 can be reduced to two J_1 functions with the space-time dimension shifted $D \to D-2$.

Relation between $\mathcal{R}$ and generalized hypergeometric function **[B. C Carlson,** Lauricella's hypergeometric function F_D**, Journal of Mathematical Analysis and Applications,** 7, 452-470(1963)**]**.

$$\mathcal{R}_{-a}(b_1, b_2, \cdots, b_N; \{z_i\}) = F\Big(a; b_1, b_2, \cdots, b_N; \sum_{j=1}^{N} b_j; \{1 - z_i\}\Big),$$

$$\text{with} \quad F \equiv {}_2F_1, F_1, F_D, F_S, \cdots .$$

Gauss ${}_2F_1$ hypergeometric functions representations for J_2:

$$\frac{J_2}{\Gamma(2-\frac{D}{2})} = \frac{\sqrt{\pi}\,\Gamma(\frac{D}{2}-1)}{\Gamma(\frac{D-1}{2})} \frac{\left(\overline{m}_2^2\right)^{\frac{D-2}{2}}}{2\lambda_{12}} \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1 - m_1^2/\overline{m}_2^2}} + (1 \leftrightarrow 2)\right]$$

$$-\frac{\Gamma(\frac{D}{2}-1)}{\Gamma(\frac{D}{2})} \left\{ \left(\frac{\partial_2 \lambda_{12}}{2\lambda_{12}}\right) \frac{(m_1^2)^{\frac{D-2}{2}}}{\sqrt{1 - m_1^2/\overline{m}_2^2}}\; {}_2F_1\left[\begin{matrix} \frac{D-2}{2}, \frac{1}{2}; \\ \frac{D}{2}; \end{matrix} \; \frac{m_1^2}{\overline{m}_2^2}\right] + (1 \leftrightarrow 2) \right\},$$

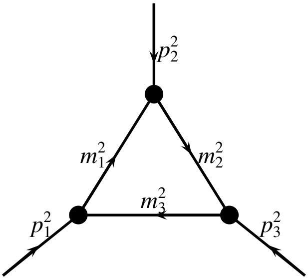
with $\lambda_{12} = \lambda(p^2, m_1^2, m_2^2)$, $\overline{m}_2^2 = -\dfrac{\lambda_{12}}{4p^2}$ and $\partial_i = \dfrac{\partial}{\partial m_i^2}$ for $i = 1, 2$. This is totally in agreement with Eq. (53) of **[J. Fleischer et al, Nucl. Phys. B 672 (2003) 303]**.

The integral J_3 takes the form of

$$
\begin{aligned}
J_3 &= \frac{\pi^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_\perp \frac{l_\perp^{D-3}}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3}, \\
\mathcal{P}_1 &= (l_0+q_{10})^2 - l_1^2 - l_\perp^2 - m_1^2 + i\rho, \\
\mathcal{P}_2 &= (l_0+q_{20})^2 - (l_1+q_{21})^2 - l_\perp^2 - m_2^2 + i\rho, \\
\mathcal{P}_3 &= l_0^2 - l_1^2 - l_\perp^2 - m_3^2 + i\rho.
\end{aligned}
$$

Partition for the integrand of J_3 is as follows

$$
\frac{1}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3} = \sum_{k=1}^{3} \frac{1}{\mathcal{P}_k \prod\limits_{\substack{l=1 \\ k\neq l}}^{3} (\mathcal{P}_l - \mathcal{P}_k)}.
$$



Taking over the $l_\perp$ integral, we get

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D-2}{2}} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \sum_{k=1}^{3} \frac{\left(-l_0^2 + l_1^2 + m_k^2 - i\rho\right)^{\frac{D}{2}-2}}{\prod_{\substack{l=1 \\ k\neq l}}^{3} (a_{lk} l_0 + b_{lk} l_1 + c_{lk})}$$

Making a shift $\tilde{l}_1 = l_1 + l_0$, J_3 is casted into the form of

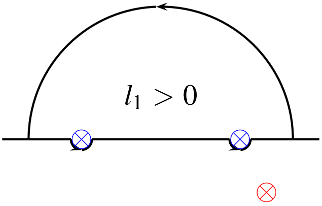
$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D-2}{2}} \sum_{k=1}^{3} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \frac{[l_1^2 - 2l_1 l_0 + m_k^2 - i\rho]^{\frac{D}{2}-2}}{\prod_{\substack{l=1 \\ k\neq l}}^{3} [AB_{lk} l_0 + b_{lk} l_1 + c_{lk}]}$$

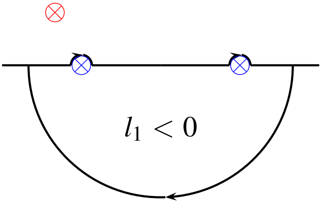
$$a_{lk} = 2(q_{l0} - q_{k0}), \qquad b_{lk} = -2(q_{l1} - q_{k1}),$$
$$c_{lk} = (q_k - q_l)^2 + m_k^2 - m_l^2, \qquad AB_{lk} = a_{lk} - b_{lk} \in \mathbb{R}.$$

It is important to note that $a_{lk}, b_{lk} \in \mathbb{R}$ and $c_{lk} \in \mathbb{C}$.

Making a shift $\tilde{l}_1 = l_1 + l_0$, J_3 is casted into the form of

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D-2}{2}} \sum_{k=1}^{3} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \frac{\left[l_1^2 - 2l_1 l_0 + m_k^2 - i\rho\right]^{\frac{D}{2}-2}}{\prod_{\substack{l=1 \\ k\neq l}}^{3} \left[AB_{lk} l_0 + b_{lk} l_1 + c_{lk}\right]}$$





$$l_0 = \frac{l_1^2 + m_k^2 - i\rho}{2l_1}, \quad \mathrm{Im}(l_0) = -\frac{m_{0k}\Gamma_k + \rho}{2l_1}.$$

After applying residue theorem, the l_0-integrations are taken

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D}{2}} i \sum_{k=1}^{3} \sum_{\substack{l=1 \\ k\neq l}}^{3} \frac{[1-\delta(AB_{lk})]}{A_{mlk}} \times$$

$$\times \left\{ f_{lk}^{+} \int_0^{\infty} dz \frac{\left[\left(1+\frac{2b_{lk}}{AB_{lk}}\right) z^2 + 2\frac{c_{lk}}{AB_{lk}} z + m_k^2 - i\rho\right]^{\frac{D}{2}-2}}{(z+F_{mlk})} \right.$$

$$\left. + f_{lk}^{-} \int_0^{\infty} dz \frac{\left[\left(1+\frac{2b_{lk}}{AB_{lk}}\right) z^2 - 2\frac{c_{lk}}{AB_{lk}} z + m_k^2 - i\rho\right]^{\frac{D}{2}-2}}{(z-F_{mlk})} \right\},$$

$$A_{mlk} = -AB_{km}\, b_{lk} + AB_{lk}\, b_{km}; \; C_{mlk} = -AB_{km}\, c_{lk} + AB_{lk}\, c_{km} \in \mathbb{C},$$

$$F_{mlk} = \frac{C_{mlk}}{A_{mlk}} \pm i\rho', \quad \rho' \to 0^{+}.$$

$$f_{lk}^{+} = \begin{cases} 2, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) > 0, \\ 1, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) = 0, \\ 0, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) < 0. \end{cases} \quad \text{and} \quad f_{lk}^{-} = \begin{cases} 2, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) < 0, \\ 1, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) = 0, \\ 0, & \text{if } \operatorname{Im}\left(-\frac{c_{lk}}{AB_{lk}}\right) > 0. \end{cases}$$

The integral J_3 can be presented in terms of $\mathcal{R}$-functions as follows

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D}{2}} i\, \mathcal{B}(4-D,1) \sum_{k=1}^{3} \sum_{\substack{l=1 \\ k\neq l}}^{3} \frac{[1-\delta(AB_{lk})]}{A_{mlk}} \times$$
$$\times \left\{ \mathcal{S}_{lk}^{+} f_{lk}^{+} \mathcal{R}_{D-4}\left(2-\frac{D}{2}, 2-\frac{D}{2}, 1; Z_{lk}^{(1)}, Z_{lk}^{(2)}, F_{mlk}\right) + \right.$$
$$\left. + \mathcal{S}_{lk}^{-} f_{lk}^{-} \mathcal{R}_{D-4}\left(2-\frac{D}{2}, 2-\frac{D}{2}, 1; -Z_{lk}^{(1)}, -Z_{lk}^{(2)}, -F_{mlk}\right) \right\},$$

$$\mathcal{S}_{lk}^{\pm} = (\alpha_{lk} - i\rho)^{\frac{D-4}{2}} \operatorname{Exp}\left[\pi i \theta\left(-\alpha_{lk}\right) \theta[\mp \operatorname{Im}(Z_{lk}^{(1)})] \theta[\mp \operatorname{Im}(Z_{lk}^{(2)})]\,(D-4)\right] \times$$
$$\times \operatorname{Exp}\left[-\pi i \theta\left(\alpha_{lk}\right) \theta[\pm \operatorname{Im}(Z_{lk}^{(1)})] \theta[\pm \operatorname{Im}(Z_{lk}^{(2)})]\,(D-4)\right].$$

Relation between $\mathcal{R}$ and generalized hypergeometric function [B. C Carlson, Lauricella's hypergeometric function F_D, Journal of Mathematical Analysis and Applications, 7, 452-470(1963)].

$$\mathcal{R}_{-a}(b_1, b_2, \cdots, b_N; \{z_i\}) = F\Big(a; b_1, b_2, \cdots, b_N; \sum_{j=1}^{N} b_j; \{1 - z_i\}\Big),$$

$$\text{with} \quad F \equiv {}_2F_1, F_1, F_D, F_S, \cdots .$$

Appell F_1 representation for J_3 is as

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D}{2}} i\, \mathcal{B}(4-D,1) \sum_{k=1}^{3} \sum_{\substack{l=1 \\ k \neq l}}^{3} \frac{[1-\delta(AB_{lk})]}{A_{mlk}} \times$$

$$\times \left[\mathcal{S}^{+}_{lk}\, f^{+}_{lk}\, (F_{mlk})^{D-4} + \mathcal{S}^{-}_{lk}\, f^{-}_{lk}\, (-F_{mlk})^{D-4}\right] \times$$

$$\times F_1\left(4-D; 2-\frac{D}{2}, 2-\frac{D}{2}; 5-D; 1-\frac{Z^{(1)}_{lk}}{F_{mlk}}, 1-\frac{Z^{(2)}_{lk}}{F_{mlk}}\right)$$

$\rightarrow$ Six Appell F_1 functions **[J. Fleischer et al, Nucl. Phys. B 672 (2003) 303]**.

The calculation

One-loop four-point functions ($D = 4 - 2\varepsilon$ at ε^0-expansion)

$$\frac{J_4}{i\pi^2} = \sum_{k=1}^{4} \sum_{\substack{l=1\\ k\neq l}}^{4} \sum_{\substack{m=1\\ m\neq l\\ m\neq k}}^{4} \frac{\Big(1-\delta(AC_{lk})\Big)\Big(1-\delta(B_{mlk})\Big)}{AC_{lk}\left(B_{mlk}A_{nlk} - B_{nlk}A_{mlk}\right)} \times$$

$$\times \Bigg[\int_0^{\infty} dz\, G(z) \Bigg\{ (f^+_{lk}\, g^+_{mlk} + f^-_{lk}\, g^+_{mlk}) \ln\left(\frac{F_{nmlk}}{\beta_{mlk}}\right) - f^+_{lk}\, g^+_{mlk} \ln\left(\frac{z+F_{nmlk}}{\beta_{mlk}}\right)$$

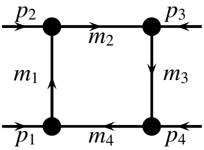
$$- f^+_{lk}\, g^-_{mlk} \ln\left(-\frac{z+F_{nmlk}}{\beta_{mlk}}\right) - (f^-_{lk}\, g^+_{mlk} + f^+_{lk}\, g^+_{mlk}) \ln\left(\frac{S(\sigma_{mlk}, z)}{P_{mlk}z + Q_{mlk}}\right)$$

$$+ f^+_{lk}\, g^+_{mlk} \ln\left(\frac{S(\sigma_{mlk}=0, z)}{P_{mlk}z + Q_{mlk}}\right) + f^+_{lk}\, g^-_{mlk} \ln\left(-\frac{S(\sigma_{mlk}=0, z)}{P_{mlk}z + Q_{mlk}}\right) \Bigg\}$$

$$+ \int_{-\infty}^{0} dz\, G(z) \Bigg\{ -f^+_{lk}\, g^-_{mlk} \ln\left(-\frac{F_{nmlk}}{\beta_{mlk}}\right) + (f^-_{lk}\, g^-_{mlk} + f^-_{lk}\, g^+_{mlk}) \ln\left(\frac{z+F_{nmlk}}{\beta_{mlk}}\right)$$

$$- f^-_{lk}\, g^-_{mlk} \ln\left(\frac{F_{nmlk}}{\beta_{mlk}}\right) - (f^-_{lk}\, g^+_{mlk} + f^-_{lk}\, g^-_{mlk}) \ln\left(\frac{S(\sigma_{mlk}=0, z)}{P_{mlk}z + Q_{mlk}}\right)$$

$$+ f^-_{lk}\, g^-_{mlk} \ln\left(\frac{S(\sigma_{mlk}, z)}{P_{mlk}z + Q_{mlk}}\right) + f^+_{lk}\, g^-_{mlk} \ln\left(-\frac{S(\sigma_{mlk}, z)}{P_{mlk}z + Q_{mlk}}\right) \Bigg\} \Bigg].$$



$$
\begin{aligned}
G(z) &= \frac{1}{Z_{mlk}\ z^2 + (E_{mlk}\beta_{mlk} - Q_{mlk} - P_{mlk}F_{nmlk})z - \beta_{mlk}(m_k^2 - i\rho) - F_{nmlk}Q_{mlk}}, \\
S(\sigma_{mlk}, z) &= (D_{mlk} + P_{mlk}\sigma_{mlk})z^2 + (E_{mlk} + Q_{mlk}\sigma_{mlk})z - m_k^2 + i\rho,
\end{aligned}
$$

The basics integrals:

$$
\begin{aligned}
\mathcal{R}_1(x,y) &= \int_0^\infty \frac{1}{(z+x)(z+y)} dz = \frac{\ln(x) - \ln(y)}{x-y}, \\
\mathcal{R}_2(r,x,y) &= \int_0^\infty \frac{\ln(1+rz)}{(z+x)(z+y)} dz = -\frac{1}{x-y}\Big[\mathrm{Li}_2(1-rx) - \mathrm{Li}_2(1-ry)\Big] \\
&\quad -\frac{1}{x-y}\Big[\eta(x,r)\ln(1-rx) - \eta(y,r)\ln(1-ry)\Big],
\end{aligned}
$$

with $r, x, y \in \mathbb{C}$.

Numerical results (ε^0-expansion)

FORTRAN and MATHEMATICA programs

Syntax of these functions: $\text{ONELOOPNPT}(D; p_i p_j; m_i^2; \rho)$,

$\text{ONELOOP2PT}(D = 4 - 2\varepsilon; p^2, 10 - i, 20 - 2i, \rho = 10^{-15})$

p^2	This work LoopTools [T. Hahn, Comput. Phys. Commun. **118** (1999) 153]
100	$-1.8207774022530800 + 1.9494059717871977\ i$ $-1.8207774022530803 + 1.9494059717871979\ i$
-100	$-3.4154066334121885 + 0.0503812303761450\ i$ $-3.4154066334121893 + 0.0503812303761446\ i$

$\text{ONELOOP3PT}(D = 4 - 2\varepsilon; p_1^2, p_2^2, p_3^2; 10 - 3\ i, 20 - 4\ i, 30 - 5\ i, 10^{-15})$

(p_1^2, p_2^2, p_3^2)	This work LoopTools
$(100, 200, -300)$	$0.0003021179436319 26 - 0.0221758348170128 30\ i$ $0.0003021179436319 17 - 0.0221758348170128 31\ i$
$(100, -200, -300)$	$-0.0122747309297076 54 - 0.0052536300737299 39\ i$ $-0.0122747309297076 52 - 0.0052536300737299 34\ i$
$(-100, -2000, -300)$	$-0.0033329053588211 72 - 0.0001461090202180 81\ i$ $-0.0033329053588211 72 - 0.0001461090202180 78\ i$

Numerical results (ε^0-expansion)

ONELOOP4PT$(D = 4 - 2\varepsilon; p_1^2, p_2^2, p_3^2 p_4^2, s, t; 10, 20, 30, 40; 10^{-15})$

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	(This work) $\times 10^{-4}$ (LoopTools) $\times\ 10^{-4}$
$(-10, 60, 10, 90, 200, 5)$	$-11.268763555152268 + 14.171199111711949\ i$ $-11.268763555152266 + 14.171199111711946\ i$
$(-10, -60, -10, -90, 200, -5)$	$2.0640147463938176 + 2.1335567055149013\ i$ $2.0640147463938226 + 2.1335567055149037\ i$
$(-10, -60, -25, -90, -20, -5)$	$1.5152693508494305 + \mathcal{O}(10^{-21})\ i$ $1.5152693508494312 + \mathcal{O}(10^{-19})\ i$

ONELOOP4PT$(D = 4 - 2\varepsilon; -10, -70, -20, -100, -15, -5; \{m_i^2\}; 10^{-15})$

$(m_1^2, m_2^2, m_3^2, m_4^2)$	(This work) $\times 10^{-4}$ (LoopTools) $\times 10^{-4}$
$(10 - 2\ i, 20, 30 - 3\ i, 40)$	$1.4577467887809371 + 0.13004659190213070\ i$ $1.4577467887809479 + 0.13004659190214078\ i$
$(10 - 2\ i, 20, 30 - 3\ i, 80)$	$1.0235403166014101 + 0.0874193853007884\ i$ $1.0235403166014069 + 0.0874193853007809\ i$
$(10 - 2\ i, 20, 30 - 3\ i, 120 - 10\ i)$	$0.79634677966095624 + 0.1085714569206661\ i$ $0.79634677966095526 + 0.1085714569206717\ i$

Numerical results (Higher-power ε-expansion: preliminary)

ONELOOP3PT$(9.75-2\varepsilon;\ -100, -200, -300;\ 100, 200, 300;\ 10^{-15})$

ε^n	This work	AMBRE/MB
ε^0	-74443.385565622551823	$-74443.7 \pm 2.$
ε^1	$+871806.32839622303446$	$+871851. \pm 93.$
ε^2	$-7.5991779166133152178 \cdot 10^6$	$(-7.593 \pm 0.004) \cdot 10^6$
ε^3	$+6.1824473581125076427 \cdot 10^7$	$(+6.15 \pm 0.01) \cdot 10^7$
ε^4	$-4.9601566778927055677 \cdot 10^8$	$(-4.92 \pm 0.04) \cdot 10^8$

AMBRE/MB: [**J. Gluza et al, Comput. Phys. Commun. 177 (2007) 879**].

Conclusion

Conclusion

- The detailed calculations for scalar one-loop integrals with complex internal masses have presented.
- The analytic results for two-, and three- point functions have written in terms of Carlson's function as well as the generalized hypergeometric series (general space-time dimension D).
- The analytic results for four-point functions have expressed in terms of logarithm and dilogarithm functions ($D = 4 - 2\varepsilon$ at ε^0-expansion).
- A package for numerical evaluations for one-loop written in `MATHEMATICA` and `FORTRAN` have built and have compared with `LoopTools` and `AMBRE/MB`.

Outlook

- Scalar one-loop four-point function with complex internal masses in general space-time dimension.
- Tensor one-loop integrals ...
- Two-loop integrals ...

Thank you very much for your attention!