Using the Newtonian gauge to solve spherically symmetric problems in general relativity

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Overview

- What is the Newtonian gauge (NG)?
- Basis of theory behind the NG
- Comparison with the comoving gauge
- Advantages of the NG
- Modelling expanding underdensities (voids)
- Modelling the Local Group as a collapsing overdensity

Newtonian gauge (NG)

- NOT the commonly known, perturbed form of FRW!
- Name reflects Newtonian nature of physics
- Can be used to solve general spherically symmetric forms of the Einstein equations
- Based on a gauge theory of gravity – Lasenby, Doran and Gull, 1998 (arxiv 0405033) – using geometric algebra
    - Can be translated into metric-based algebra, using 'tetrads'

Tetrad-based method

- Coordinate basis vectors $\boldsymbol{e}_\mu$

$$\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = g_{\mu\nu}$$

- Local Lorentz frame basis vectors $\hat{\boldsymbol{e}}_i$

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$$

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Tetrads

$$\hat{\boldsymbol{e}}_k = e_k{}^\mu \boldsymbol{e}_\mu,$$
$$\boldsymbol{e}_\mu = e^k{}_\mu \hat{\boldsymbol{e}}_k.$$

Tetrad-based method

- Spherically symmetric ⇒ just four functions
- Can choose one of them to be zero by using the invariance of GR under general coordinate transformations
- Newtonian gauge – tetrads can be represented by functions $f_1(r,t)\,,\ g_1(r,t)\,,\ g_2(r,t)$

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- Metric: $$ds^2 = \frac{g_1^2 - g_2^2}{f_1^2 g_1^2}dt^2 + \frac{2g_2}{f_1 g_1^2}dr\,dt - \frac{1}{g_1^2}dr^2 - r^2 d\Omega^2$$

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- A comoving observer has four-velocity

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Fluid velocity

Comoving gauge (CG)

- General spherically symmetric comoving coordinates

$$ds^2 = A(r', t')^2 dt'^2 - B(r', t')^2 dr'^2 - R(r', t')^2 d\Omega^2$$

- The Lemaître-Tolman-Bondi (LTB) metric is a subset, with

  $A(r', t') = 1$

  - But only works with zero pressure

- Most popular way of representing spherical inhomogeneities
- An extension of FRW metric
- A comoving observer is stationary in these coordinates

$$\begin{aligned}[u'^\mu] = (\dot{t}', \dot{r}', \dot{\theta}', \dot{\phi}') &= (1/A, 0, 0, 0) \\ &= (1, 0, 0, 0) \qquad \text{for LTB}\end{aligned}$$

Gauge transformation

- NG: $$ds^2 = \frac{g_1^2 - g_2^2}{f_1^2 g_1^2} dt^2 + \frac{2g_2}{f_1 g_1^2} dr\, dt - \frac{1}{g_1^2} dr^2 - r^2 d\Omega^2$$

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$$t = t', \quad r = R(r', t'), \quad \text{where} \quad \frac{\partial R}{\partial t'} = \frac{g_2}{f_1}$$

- However, $\frac{\partial R}{\partial r'}$ *remains unknown* – residual gauge freedom in the comoving gauge.

Comoving gauge:

$$ds^2 = A(r', t')^2 dt'^2 - B(r', t')^2 dr'^2 - R(r', t')^2 d\Omega^2$$

Gauge transformation

- Physical interpretation:

$$\frac{\partial}{\partial t'} = \frac{1}{f_1}\left(f_1 \frac{\partial}{\partial t} + g_2 \frac{\partial}{\partial r}\right)$$

Convective derivative

- Comoving gauge: Lagrangian picture (following the fluid particles)
- NG: Eulerian picture

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- Equations often have Newtonian analogues, yet fully consistent with GR
  - E.g. with dust,

$$\left(\frac{\partial}{\partial t} + g_2 \frac{\partial}{\partial r}\right) g_2 = -\frac{M}{r^2} + \tfrac{1}{3}\Lambda r$$

Euler equation

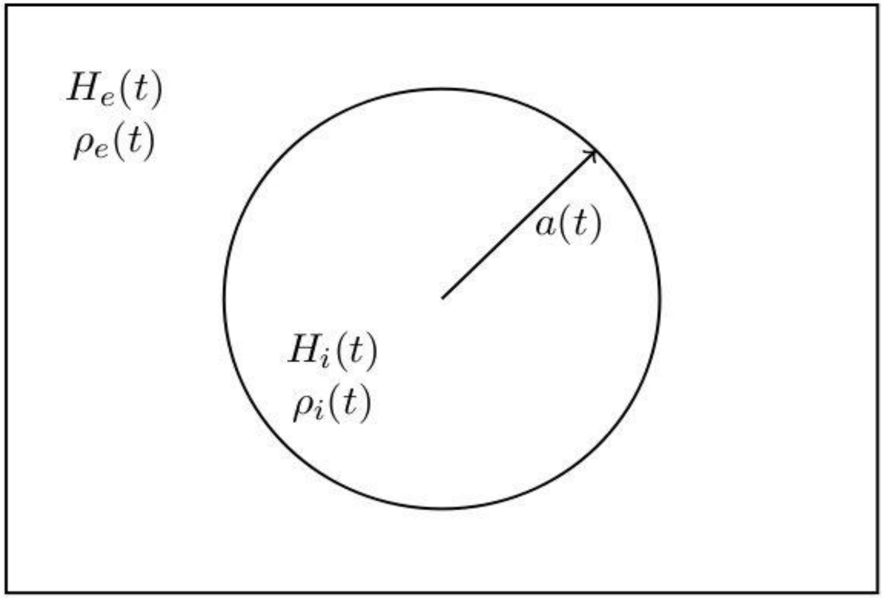
$$\tfrac{1}{2} g_2 - M/r = \tfrac{1}{2}\left(g_1^2 - 1\right)$$

Bernoulli equation

Advantages of the NG

- Can offer additional insights when looking at black holes
  - Time-reversal asymmetry

Model



$$M = \begin{cases} \frac{4}{3}\pi\rho_i(t)r^3, & r \leq a(t), \\ \frac{4}{3}\pi\rho_e(t)r^3 + m(t), & r > a(t), \end{cases}$$

where $m(t) \equiv \frac{4}{3}\pi(\rho_i(t) - \rho_e(t))a(t)^3$

- Based on model explored by Nandra, Hobson and Lasenby (arxiv 1307.0526)
- $H_i(t)$ and $H_e(t)$ can be chosen to have any form – determines velocity field at all times
  - Determines pressure profile

Model

- Aim is to calculate effect of the inhomogeneity on an incoming photon
- Model exterior to tend to ΛCDM. $H_e(t) = \dfrac{a_e'(t)}{a_e(t)}$
- $$a_e'^2(t) = H_e^2(t_0)\left(\Omega_{m,e}a_e^{-1}(t) + \Omega_{\Lambda,e}a_e^2(t) + \Omega_{k,e}\right)$$

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- Parameterised $H_i(t)$ as $H_i(t) = \frac{a_i'(t)}{a_i(t)}$
  $$a_i'^2(t) = H_i^2(t_0)\left(\Omega_{m,i}a_i^{-1}(t) + \Omega_{\Lambda,i}a_i^2(t) + \Omega_{k,i}\right)$$
- Integrate photon momentum equations ⇒ find relationship between d_A and z

Model

- Find perceived value of $H(t_0)$ if we adopt an FRW cosmology:

$$d_A = \frac{c}{(1+z)H_{\text{obs}}(t_0)} \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0}}}$$

- Compare this with value of $H_e(t_0)$

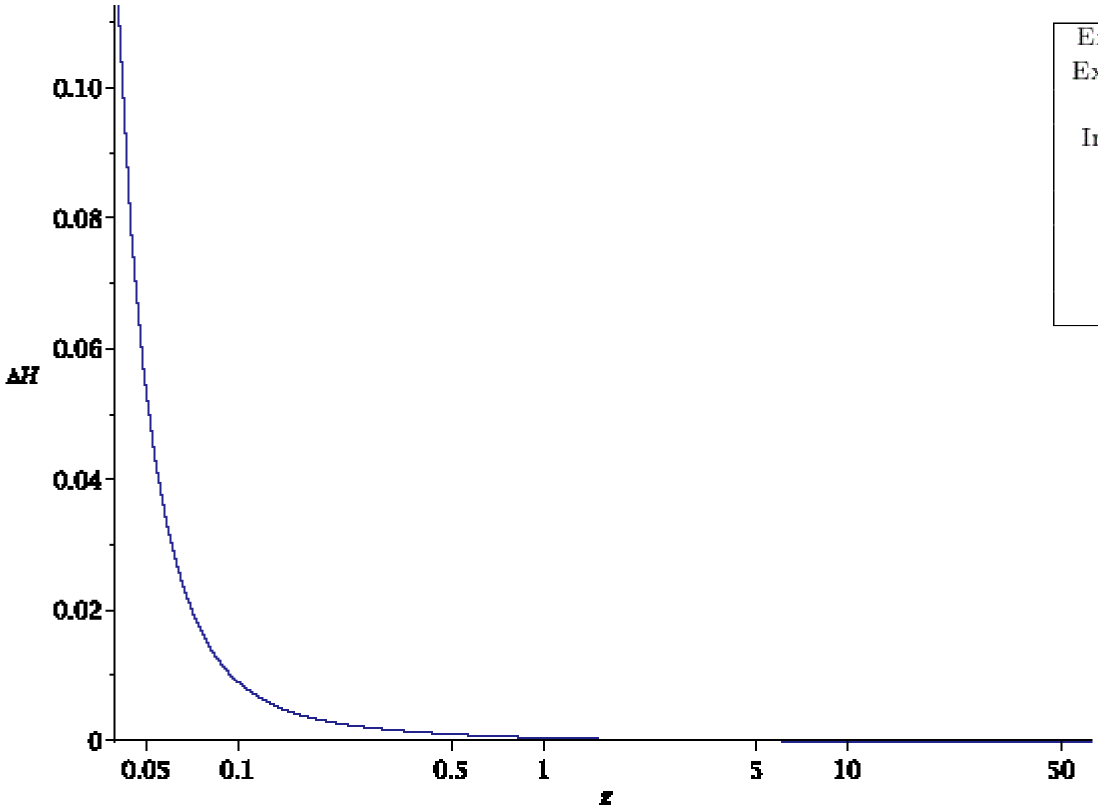
$$\Delta H = \frac{H_{\text{obs}}(t_0) - H_e(t_0)}{H_e(t_0)}$$

- Uniform $\rho(t) \Rightarrow$ analytic solutions for pressure (which has never been done before)
  - Verified pressure has little effect on photon
- Spatially varying ρ and its effect on anisotropy on CMB have been explored using the NG
  - See Lasenby et al, arxiv 9810123

Expanding void

- Expanding voids have been under a lot of interest recently
  - Cold spot in CMB - Szapudi et al ,2014 (arxiv 1406.3622)
  - Alternative to Λ- Alnes et al, 2006 (0512006)
  - Effect on H_0 – Romano & Vallejo, 2014 (1403.2034)
  - Normally use LTB metric
- Tried adopting our model to a void of this size

Expanding void



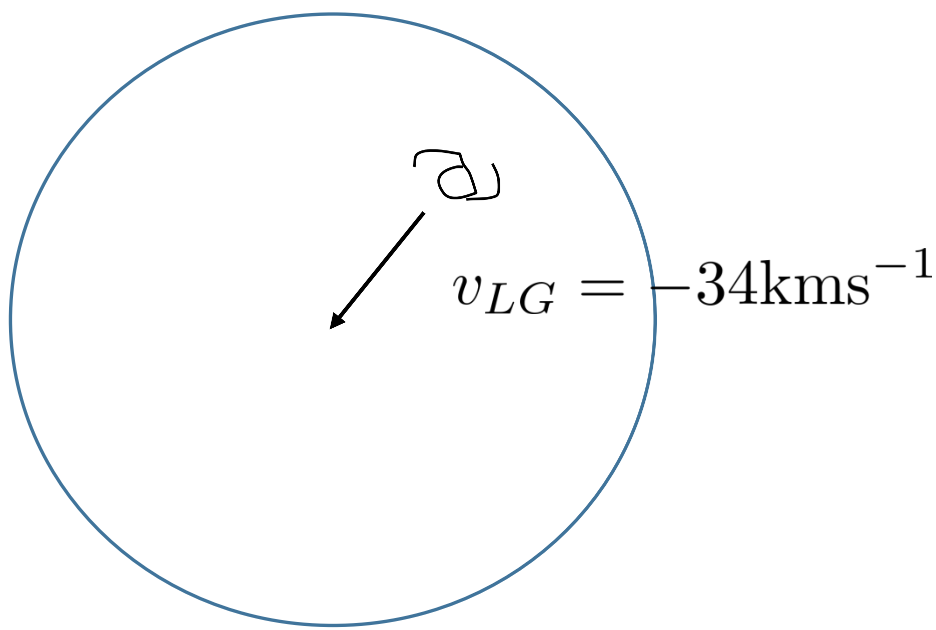
Exterior Hubble parameter	$H_e(t_0)$	$72\ \mathrm{km\,s^{-1}Mpc^{-1}}$
Exterior density parameters	$\Omega_{m,e}$	0.3
	$\Omega_{\Lambda,e}$	0.7
Interior Hubble parameter	$H_i(t_0)$	80
Interior parameters	$\Omega_{m,i}$	0.2
	$\Omega_{\Lambda,i}$	0.8
	$\Omega_{k,i}$	0
Age of universe	t_0	13.1 Gyr
Radius	$a(t_0)$	150 Myr

- Similar void to those explored by Romano & Vallejo , 2014 (1403.2034)

- At $z = 0.04,\ \Delta H \approx 11\%$
- Similar magnitudes to those obtained by Romano & Vallejo

Collapsing Local Group

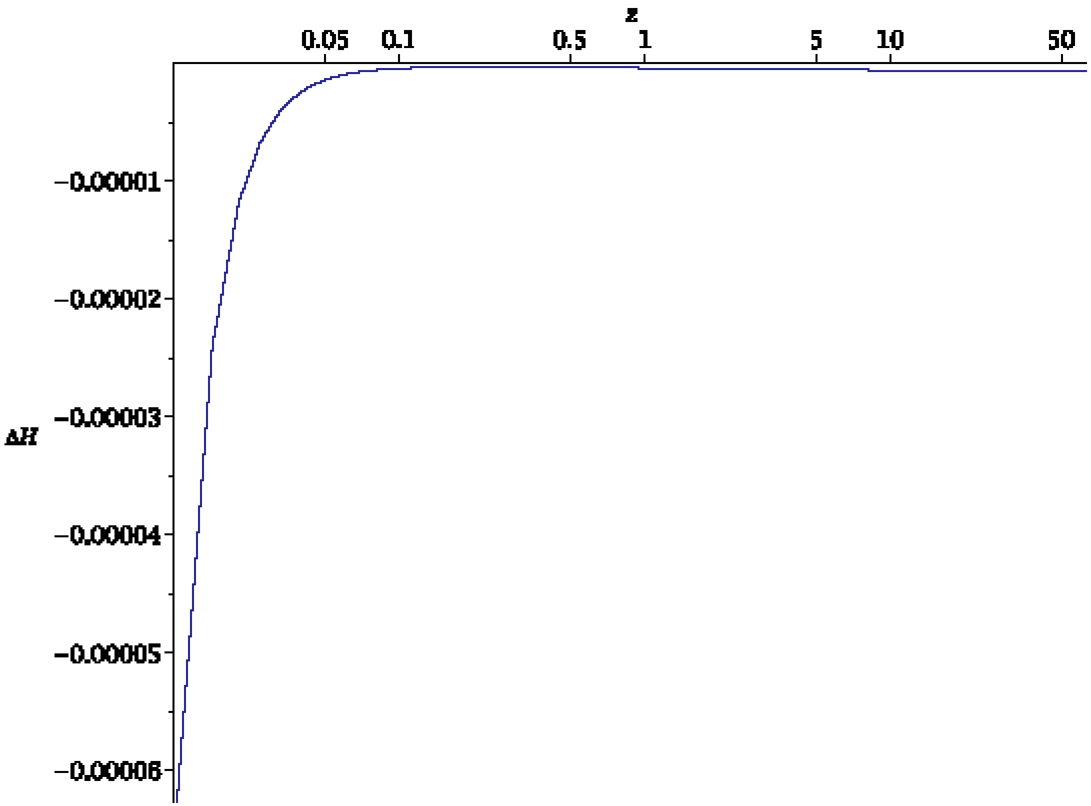
- We know we are in an inhomogeneity – the Local Group!
- Long known that Andromeda (M31) has been moving towards us



Collapsing Local Group

- Recently, the proper motion of M31 has been measured
  - Will merge with MW in 5.86 Gyr. (van der Marel et al, 2012,arxiv 1205.6865)
- Can constrain $H_i(t)$ using the collapse time, and by assuming M31 is following the Hubble flow

Collapsing Local Group

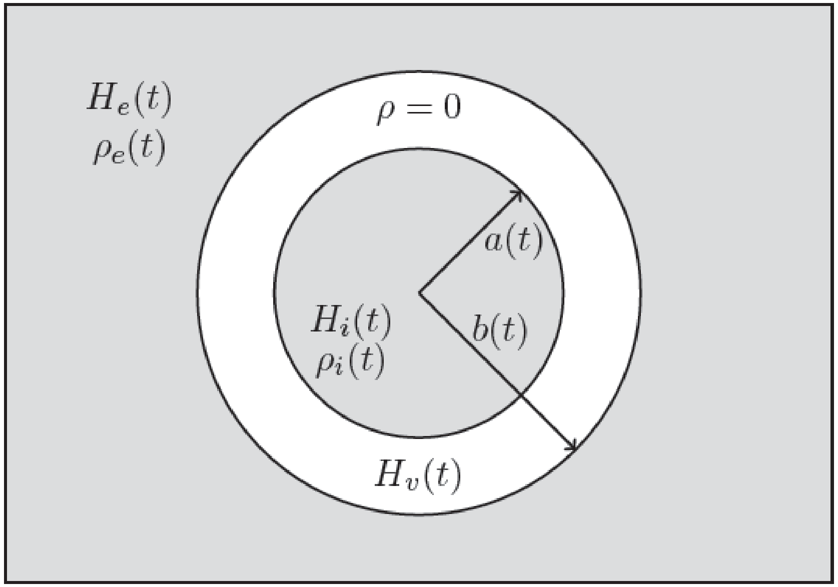


Exterior Hubble parameter	$H_e(t_0)$	72
Interior Hubble parameter	$H_i(t_0)$	-87
Exterior density parameters	$\Omega_{m,e}$	0.3
	$\Omega_{\Lambda,e}$	0.7
Interior parameters	$\Omega_{m,i}$	1.08
	$\Omega_{\Lambda,i}$	-2.40
	$\Omega_{k,i}$	2.32
Age of universe	t_0	13.1 Gyr
Reception time	t_R	13.1 Gyr
Radius	$a(t_0)$	1.55 Myr

- Small effect
- $z = 0.04,\ \Delta H \approx -2 \cdot 10^{-4}\%$

Other applications of NG …

- Can use more general velocity fields and density to model Local Group
- Have also been looking at claims made by Melia & Shevchuk, 2012 (arxiv 1206.6527) – the 'R_h=*ct*' theory
  - Inconsistent with CMB fluctuations, structure formation
  - Can use the NG to reproduce their set-up



- Newtonian gauge can provide new insights in spherically symmetric systems