Using the Newtonian gauge to solve spherically symmetric problems in general relativity

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Overview

- What is the Newtonian gauge (NG)?
- Basis of theory behind the NG
- Comparison with the comoving gauge
- Advantages of the NG
- Modelling expanding underdensities (voids)
- Modelling the Local Group as a collapsing overdensity
Newtonian gauge (NG)

- NOT the commonly known, perturbed form of FRW!
- Name reflects Newtonian nature of physics
- Can be used to solve general spherically symmetric forms of the Einstein equations
- Based on a gauge theory of gravity – Lasenby, Doran and Gull, 1998 (arxiv 0405033) – using geometric algebra
  - Can be translated into metric-based algebra, using ‘tetrads’
Tetrad-based method

- Coordinate basis vectors $e_\mu$
  \[ e_\mu \cdot e_\nu = g_{\mu\nu} \]

- Local Lorentz frame basis vectors $\hat{e}_i$
  \[ \hat{e}_i \cdot \hat{e}_j = \eta_{ij} = \text{diag}(1, -1, -1, -1) \]
Tetrad-based method

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Tetrads

\[ \hat{e}_k = e^\mu_k e_\mu, \]

\[ e_\mu = e^k_\mu \hat{e}_k. \]
Tetrad-based method

- Spherically symmetric $\Rightarrow$ just four functions
- Can choose one of them to be zero by using the invariance of GR under general coordinate transformations
- Newtonian gauge – tetrads can be represented by functions $f_1(r, t), g_1(r, t), g_2(r, t)$
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- Metric:

$$ds^2 = \frac{g_1^2 - g_2^2}{f_1^2 g_1^2} dt^2 + \frac{2g_2}{f_1 g_1^2} dr dt - \frac{1}{g_1^2} dr^2 - r^2 d\Omega^2$$
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Metric:

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- A comoving observer has four-velocity

$$[u^\mu] \equiv [\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}] = [f_1, g_2, 0, 0]$$
Tetrad-based method

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  Fluid velocity
Comoving gauge (CG)

- General spherically symmetric comoving coordinates
  \[ ds^2 = A(r', t')^2 dt'^2 - B(r', t')^2 dr'^2 - R(r', t')^2 d\Omega^2 \]
- The Lemaître-Tolman-Bondi (LTB) metric is a subset, with
  \[ A(r', t') = 1 \]
  - But only works with zero pressure
- Most popular way of representing spherical inhomogeneities
- An extension of FRW metric
- A comoving observer is stationary in these coordinates
  \[ [u'\mu] = (\dot{t}', \dot{r}', \dot{\theta}', \dot{\phi}') = \left( \frac{1}{A}, 0, 0, 0 \right) \]
  \[ = \left( 1, 0, 0, 0 \right) \] for LTB
Gauge transformation

- NG:

\[ ds^2 = \frac{g_1^2 - g_2^2}{f_1^2 g_1^2} dt^2 + \frac{2g_2}{f_1 g_1^2} dr dt - \frac{1}{g_1^2} dr^2 - r^2 d\Omega^2 \]

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\[ t = t', \quad r = R(r', t'), \quad \text{where} \quad \frac{\partial R}{\partial t'} = \frac{g_2}{f_1} \]

However, \( \frac{\partial R}{\partial r'} \) remains unknown – residual gauge freedom in the comoving gauge.

Comoving gauge:
\[ ds^2 = A(r', t')^2 dt'^2 - B(r', t')^2 dr'^2 - R(r', t')^2 d\Omega^2 \]
Gauge transformation

- Physical interpretation:

\[ \frac{\partial}{\partial t'} = \frac{1}{f_1} \left( f_1 \frac{\partial}{\partial t} + g_2 \frac{\partial}{\partial r} \right) \]

Convective derivative

- Comoving gauge: Lagrangian picture (following the fluid particles)
- NG: Eulerian picture
Advantages of the NG

- Physical quantities easily extracted
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  - In LTB, need to fix gauge freedom by setting arbitrary initial conditions – in some cases by making the density homogeneous (Romano & Chen, arxiv 1104.0730), which is very unintuitive!
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- Equations often have Newtonian analogues, yet fully consistent with GR
  - E.g. with dust,
    \[
    \left( \frac{\partial}{\partial t} + g_2 \frac{\partial}{\partial r} \right) g_2 = -\frac{M}{r^2} + \frac{1}{3} \Lambda r
    \]
    Euler equation

\[
\frac{1}{2} g_2 - \frac{M}{r} = \frac{1}{2} \left( g_1^2 - 1 \right)
\]
    Bernoulli equation
Advantages of the NG

- Can offer additional insights when looking at black holes
  - Time-reversal asymmetry
Based on model explored by Nandra, Hobson and Lasenby (arxiv 1307.0526)

- $H_i(t)$ and $H_e(t)$ can be chosen to have any form – determines velocity field at all times
- Determines pressure profile

$$M = \begin{cases} 
\frac{4}{3} \pi \rho_i(t) r^3, & r \leq a(t), \\
\frac{4}{3} \pi \rho_e(t) r^3 + m(t), & r > a(t),
\end{cases}$$

where $m(t) \equiv \frac{4}{3} \pi (\rho_i(t) - \rho_e(t)) a(t)^3$
Model

- Aim is to calculate effect of the inhomogeneity on an incoming photon
- Model exterior to tend to $\Lambda$CDM.

\[ H_e(t) = \frac{a'_e(t)}{a_e(t)} \]

\[ a'^2_e(t) = H_e^2(t_0) \left( \Omega_{m,e} a_e^{-1}(t) + \Omega_{\Lambda,e} a_e^2(t) + \Omega_{k,e} \right) \]
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- Parameterised $H_i(t)$ as \[ H_i(t) = \frac{a'_i(t)}{a_i(t)} \]
  \[ a'_i^2(t) = H_i^2(t_0) \left( \Omega_{m,i}a_i^{-1}(t) + \Omega_{\Lambda,i}a_i^2(t) + \Omega_{k,i} \right) \]
- Integrate photon momentum equations $\Rightarrow$ find relationship between $d_A$ and $z$
Model

• Find perceived value of $H(t_0)$ if we adopt an FRW cosmology:

$$d_A = \frac{c}{(1 + z)H_{\text{obs}}(t_0)} \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1 + z')^3 + \Omega_{\Lambda,0}}}$$

• Compare this with value of $H_e(t_0)$

$$\Delta H = \frac{H_{\text{obs}}(t_0) - H_e(t_0)}{H_e(t_0)}$$

• Uniform $\rho(t) \Rightarrow$ analytic solutions for pressure (which has never been done before)
  • Verified pressure has little effect on photon

• Spatially varying $\rho$ and its effect on anisotropy on CMB have been explored using the NG
  • See Lasenby et al, arxiv 9810123
Expanding void

- Expanding voids have been under a lot of interest recently
  - Cold spot in CMB - Szapudi et al, 2014 (arxiv 1406.3622)
  - Alternative to $\Lambda$ - Alnes et al, 2006 (0512006)
  - Effect on $H_0$ – Romano & Vallejo, 2014 (1403.2034)
  - Normally use LTB metric
- Tried adopting our model to a void of this size
Expanding void

- Similar void to those explored by Romano & Vallejo, 2014 (1403.2034)

- At $z = 0.04$, $\Delta H \approx 11\%$

- Similar magnitudes to those obtained by Romano & Vallejo

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<th>Parameter</th>
<th>Exterior Hubble parameter</th>
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<td>$a(t_0)$</td>
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Collapsing Local Group

- We know we are in an inhomogeneity – the Local Group!
- Long known that Andromeda (M31) has been moving towards us

\[ v_{LG} = -34 \text{km s}^{-1} \]
Collapsing Local Group

- Recently, the proper motion of M31 has been measured
  - Will merge with MW in 5.86 Gyr. (van der Marel et al, 2012, arxiv 1205.6865)

- Can constrain $H_i(t)$ using the collapse time, and by assuming M31 is following the Hubble flow
Collapsing Local Group

- Small effect
- \( z = 0.04, \Delta H \approx -2 \cdot 10^{-4} \% \)
Other applications of NG …

- Can use more general velocity fields and density to model Local Group
- Have also been looking at claims made by Melia & Shevchuk, 2012 (arxiv 1206.6527) – the ‘$R_h=ct$’ theory
  - Inconsistent with CMB fluctuations, structure formation
  - Can use the NG to reproduce their set-up
- Newtonian gauge can provide new insights in spherically symmetric systems