Pseudo-Weyl Gravity (TWG)

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It is an old dream in theoretical physics on the possibility that at very short distances there is no memory of individual particle masses, so that the theory is conformal.

Indeeed, conformal symmetry and supersymmetry are the only known symmetries that forbid the cosmological constant.

Many guises of Conformal invariance

Flat space: Dilatation symmetry (Callan-Symanzik)

In flat space, a *scale transformation* is defined as

$$x'_\mu = \lambda x_\mu \tag{1.1}$$

Scale transformations belong to the *conformal group*, $SO(2,n)$, which includes besides the *special conformal transformations*

$$x'_\mu = \frac{x^\mu - a^\mu x^2}{1 - 2a.x + a^2 x^2} \tag{1.2}$$

as well as the whole Poincaré group. The special conformal transformations C are a combination of a translation

$$T_a : \; x^\mu \to x^\mu + a^\mu \tag{1.3}$$

and an inversion

$$I : x^\mu \to -\frac{x^\mu}{x^2} \tag{1.4}$$

namely

$$C = T_a \circ I \circ T_a \tag{1.5}$$

Altogether, there are 15 parameters in the conformal group. The infinitesimal generators can be chosen [37] as

$$M_{\mu\nu} = M_{\mu\nu} \quad M_{65} = D$$
$$M_{5\mu} = \frac{1}{2}(P_\mu - K_\mu) \quad J_{6\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{1.6}$$

where D is the generator of dilatations, K_μ generate the special conformal transformations and P_μ are the ordinary translations. M_{AB} $A = 1\ldots 6$ are the generators of $SO(2,4)$.

It is always possible to improve the energy-momentum tensor

$$T \equiv g^{\mu\nu} \frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 0$$

A working definition of cosmological constant is

$$\Lambda \equiv g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = \left(1 - \frac{n}{2}\right) R - n\lambda\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} = -n\lambda = 0$$

When the spacetime metric gets position-dependent this concept does not generalize easily.

The closest we can get: local rescalings of the metric

$$g'_{\mu\nu}(x) \equiv \Omega^2(x) g_{\mu\nu}(x)$$

We want to study conformal properties in the presence of dynamical gravity

$$\sqrt{|\tilde{g}|}\tilde{R} = \sqrt{|g|}\left[\Omega^{n-2}R + (n-1)(n-2)\Omega^{n-4}(\nabla\Omega)^2\right]$$

Promote the Weyl parameter to a new graviscalar field

$$\Omega \equiv \frac{1}{M_p}\sqrt{\frac{(n-2)}{4(n-1)}}\ \phi_g^{\frac{2}{n-2}}$$

$$M_p^{n-2} \equiv \frac{1}{16\pi G_n}$$

The resulting theory is TWG, with pseudo-Weyl symmetry

$$S_{TWG} = \int d(vol) \left(-\frac{n-2}{8(n-1)} R \, \phi_g^2 - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_g \nabla_\nu \phi_g \right)$$

$$d(vol) \equiv \sqrt{|g|} d^n x$$

(Dirac, Englert et al)

Pseudo Weyl symmetry

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{\phi}_g = \Omega^{\frac{2-n}{2}} \, \phi_g$$

Why pseudo? Because one of the fields is an spurion than can be eliminated through a field redefinition

It is actually possible to work in the Einstein frame

$$G_{\mu\nu} \equiv \frac{1}{M_p^2}\left(\frac{n-2}{8(n-1)}\right)^{\frac{2}{n-2}} \phi_g^{\frac{4}{n-2}} g_{\mu\nu}$$

The Einstein metric is a singlet
(inert under Weyl transformations)

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{\phi}_g = \Omega^{\frac{2-n}{2}} \phi_g$$

The action then reduces to
EInstein-Hilbert

$$S = -M_p^{n-2} \int \sqrt{G}\, d^n x\, R[G]$$

Phases of Weyl gauge symmetry

There are two Weyl gauge orbits

Isolated point.
Symmetric Phase

$$\phi_g = 0$$

Broken phase

$$\phi_g \in \mathcal{F} \setminus \{0\}$$

Also in GR it is sometimes possible to define a symmetric phase.

Truncation of supergravity superconformal (Kallosh et al)

In the broken phase

TWG reduces classically to GR in the gauge

$$\phi_g = \sqrt{\frac{8(n-1)}{n-2}} M_p^{\frac{n-2}{2}}$$

TWG reduces classically to unimodular gravity (UG) in the gauge

$$\phi_g + 2^{\frac{3}{2}} M_p^{\frac{n-2}{2}} \sqrt{\frac{n-1}{n-2}} g^{-\frac{n-2}{4n}} = 0$$

Our aim now is however to study the much more interesting unbroken phase, in which the vacuum expectation value of the gravitational scalar field vanishes.

Change of variables corresponding to a Weyl transformation

$$0 = \delta Z \equiv \int \mathcal{D}g_{\mu\nu} \prod_i \mathcal{D}\psi_i \int d(vol)_x \omega(x) \Big\{ -2g^{\mu\nu}(x) \frac{\delta S}{\delta g_{\mu\nu}(x)} - \frac{n-2}{2}\phi_g \frac{\delta S}{\delta \phi_g} +$$

$$+2J^{\mu\nu}(x) g_{\mu\nu}(x) - J(x)\phi_g(x) \Big\} \exp \left\{ iS[g_{\mu\nu}\phi_g] + \int d(vol)\left(J^{\mu\nu} g_{\mu\nu} + J\ \phi_g\right) \right\} ($$

Off shell Ward identities

$$\left\langle 0_+ \left| g^{\mu\nu}(x) \frac{\delta S}{\delta g_{\mu\nu}(x)} + \frac{n-2}{4}\phi_g \frac{\delta S}{\delta \phi_g} \right| 0_- \right\rangle = 0$$

(They generalize to the gravitational case the tracelessness of the energy-momentum tensor)

Define the functional integral through the Einstein frame

$$e^{iW[\bar{g}_{\mu\nu},\bar{\phi}_g]} \equiv \int \mathcal{D}g_{\mu\nu}\ \mathcal{D}\phi_g\ e^{-i\frac{1}{2}\int d^4x\ \sqrt{-g}(\partial_\mu\phi_g g^{\mu\nu}\ \partial_\nu\phi_g+\frac{1}{6}\ R\ \phi_g^2)}$$

$$e^{iW[\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu},\bar{\phi}_g]]} \equiv \int \mathcal{D}G_{\mu\nu}\ e^{\frac{i}{16\pi G}\int d^4x R[G_{\mu\nu}]}$$

‘t Hooft and Veltman effective action

$$S_\infty = \frac{1}{\pi^2(n-4)}\int d^4x\sqrt{|G|}\left(\frac{149}{2880}E_4[G] + \frac{7}{320}W_4[G] + \frac{3}{128}R[G]^2\right)$$

This constructs are point Weyl invariant in four and only in four dimensions

$$\int d(vol)\ W_4\left[\Omega^2 g_{\mu\nu}\right] = \int d(vol)\ \Omega^{n-4}\ W_4\left[g_{\mu\nu}\right]$$

$$\int d(vol)\ E_4\left[\Omega^2 g_{\mu\nu}\right] = \int d(vol)\ \Omega^{n-4}\ E_4\left[g_{\mu\nu}\right]$$

This means that there is a finite residue from the pole in the infinite piece of the effective action when performing a Weyl transformation

$$\frac{1}{n-4} \times (n-4) \to \text{finite remainder}$$

Specific TWG divergences follow the same pattern

$$S_\infty = \frac{1}{\pi^2(n-4)}\int d(vol)\left\{\frac{149}{2880}E_4 + \frac{7}{320}W_4 + \left(R - 6\frac{\nabla^2\phi_g}{\phi_g}\right)^2\right\}$$

$$\left(\tilde{\nabla}^2 - \frac{n-2}{4(n-1)}\tilde{R}\right)\left(\Omega^{-\frac{n-2}{2}}\phi\right) = \Omega^{-\frac{n+2}{2}}\left(\nabla^2 - \frac{n-2}{4(n-1)}R\right)$$

$$\left(\tilde{R} - \frac{4(n-1)}{n-2}\frac{\tilde{\nabla}^2\tilde{\phi}_g}{\tilde{\phi}_g}\right)^2 = \Omega^{-4}\left(R - \frac{4(n-1)}{n-2}\frac{\nabla^2\phi_g}{\phi_g}\right)^2$$

∴ There is a conformal anomaly in TWG given by

$$\left\langle 0_+ \left| -2g^{\mu\nu}\frac{\delta S_{TWG}}{\delta g^{\mu\nu}} - \frac{n-2}{2}\phi_g \frac{\delta S_{TWG}}{\delta \phi_g} \right| 0_- \right\rangle \equiv A_{TWG} = \frac{1}{\pi^2}\left\{ \frac{7}{320} W_4 + \left(R - 6\frac{\nabla^2 \phi_g}{\phi_g} \right)^2 \right\}$$

This is at variance with some cherished beliefs

Work is in progress to check this by a direct TWG heat kernel computation

The result may seem surprising at first sight, but it is a trivial consequence of

1.- The counterterm must be conformal invariant.

2.-The only pointwise conformal invariant in four dimensions is $\sqrt{|g|}W_4$

The only logical way out would be that either

3.- There is no counterterm, that is the theory is finite

Or else

4.-Give up diffeomorphism invariance.

Then there are pointwise conformal invariants in arbitrary dimension, such as

$$(-g)^{\frac{2}{n}}\ W_4.$$

5.- It is always possible that the formula we have used to compute the conformal anomaly does not hold for some unknown reason?

$$\left\langle g^{\mu\nu} \, \frac{\delta S}{\delta g^{\mu\nu}} \right\rangle = 2 \, \frac{\delta S_{\text{eff}}}{\delta \Omega} \bigg|_{\Omega=1}$$

Backup slides

CONFORMAL SUPERGRAVITY

$$L \equiv \sqrt{-g}\, W_{\mu\nu\rho\sigma}\, W^{\mu\nu\rho\sigma} \equiv \sqrt{|g|}\, W_4$$ + lots

$$W_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

Vanishing Weyl tensor is equivalent to conformal flatness

$$W_4 \equiv R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2.$$

Finite UV theory (Fradkin & Tseytlin)

Tension with unitarity (quartic propagator)

Einstein's 1919 theory

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

Tracefree piece of EE

$$g^{\alpha\beta} \frac{\delta}{\delta g^{\alpha\beta}} S = 0$$

The EE's Trace is recovered through the Bianchi identities

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R$$

$$(\frac{1}{2} - \frac{1}{n}) \nabla^\nu R = -\frac{\kappa^2}{n} \nabla^\nu T$$

Classically it corresponds to GR with some CC.

$$\frac{n-2}{2} R + \frac{\kappa^2}{n} T \equiv \lambda$$

$$R_{\mu\nu} - \frac{1}{2} (R - 2\lambda) g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Traceless smells unimodular…

The Unimodular Gravity (UG) action principle

$$S_U \equiv -M^{n-2}\int d^n x\, R_E \;+\; S_{matt} =$$
$$-M^{n-2}\int d^n x\, g^{\frac{1}{n}}\left(R+\frac{(n-1)(n-2)}{4n^2}\frac{g^{\mu\nu}\nabla_\mu g\,\nabla_\nu g}{g^2}\right)+S_{matt}$$

Invariant under measure preserving diffeomorphisms

Traceless equations of motion

$$R_{\mu\nu}-\frac{1}{n}\,R\,g_{\mu\nu}\equiv\; M^{2-n}\left(J^g_{\mu\nu}+J^m_{\mu\nu}\right)=$$
$$\frac{(n-2)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g\nabla_\nu g}{g^2}-\frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\nu}\right)-\frac{n-2}{2n}\left(\frac{\nabla_\mu\nabla_\nu g}{g}-\frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\nu}\right)+$$
$$+\frac{M^{2-n}}{2}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{n}\,g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\;g_{\mu\nu}\right) \tag{1.9}$$

The matter sector and a new twist on TWG

Conformal weight of matter fields

$$\tilde{\phi} \equiv \Omega^{-\lambda}\phi$$
$$\delta\phi = -\lambda\omega\phi$$

$$\bar{\psi}_i = \Omega^{-\frac{n-1}{2}}\ \psi_i$$
$$\bar{\phi}_l = \Omega^{-\frac{n-2}{2}}\ \phi_l$$
$$\bar{A}_\mu = \Omega^{-\frac{n-4}{2}}\ A_\mu$$

General matter lagrangian is of the type

$$S_{\rm matt} = \int d(vol) \left\{ \frac{1}{2}\sum_i e^\mu_a \left(\bar{\psi}_i\gamma^a D_\mu\psi_i - D_\mu\bar{\psi}_i\gamma^a\psi_i\right) + \frac{1}{2}\sum_i g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - V(\phi_i) - \right.$$
$$\left. -\frac{1}{4}\sum_a F^a_{\mu\nu}(F^a)^{\mu\nu} - i\lambda\sum_{ij} \bar{\psi}_i\gamma^a e^\mu_a A^b_\mu T^b_{ij}\psi_j + \sum_{ijk} y_{ijk}\ \bar{\psi}_i\phi_k\psi_j + \sum_{p=1}\sum_{i_1\ldots i_p} g_{i_1\ldots i_p}\phi_{i_1}\ldots\phi_{i_p} \right\}$$

Dimensionful coupling constants break conformal invariance

Kinetic terms are not gauge invariant a priori

$\therefore$ Weyl gauge field

$$\tilde{W}_\mu \equiv W_\mu + \Omega\partial_\mu\Omega^{-1} = W_\mu - \Omega^{-1}\partial_\mu\Omega \equiv W_\mu - \partial_\mu\omega$$

There is an interesting object (I,O'R,W)

$$\tilde{\mathcal{F}}_{\mu\nu}[W] \equiv \nabla_\mu W_\nu - W_\mu W_\nu + \frac{1}{2}W^2\; g_{\mu\nu}$$

Under a local Weyl transformation

$$\tilde{\mathcal{F}}_{\mu\nu}[W] = \mathcal{F}_{\mu\nu}[W] - \nabla_\mu\omega_\nu + \omega_\mu\omega_\nu - \frac{1}{2}\omega_\lambda^2 g_{\mu\nu}$$

$$\tilde{\mathcal{F}}_{\mu\nu}[W] = \mathcal{F}_{\mu\nu}[W] - \tilde{\mathcal{F}}_{\mu\nu}[\omega]$$

The Schouten tensor, a geometrical object transforms exactly in the same way

$$A_{\alpha\beta} \equiv \frac{1}{n-2}\left(R_{\alpha\beta} - \frac{1}{2(n-1)} R g_{\alpha\beta}\right)$$

$$\tilde{A}_{\mu\nu} = A_{\mu\nu} - \frac{\nabla_\mu \nabla_\nu \Omega}{\Omega} + 2\frac{\nabla_\mu \Omega \nabla_\nu \Omega}{\Omega^2} - \frac{1}{2}\frac{(\nabla\Omega)^2}{\Omega^2} g_{\mu\nu} =$$
$$A_{\mu\nu} - \nabla_\mu \omega_\nu + \omega_\mu \omega_\nu - \frac{1}{2}\omega_\lambda \omega^\lambda g_{\mu\nu} = A_{\mu\nu} - \mathcal{F}_{\mu\nu}[\omega]$$

This means that we can replace the Weyl object by the Schouten tensor

$$\mathcal{F}_{\mu\nu}[W] \to A_{\mu\nu}$$

$$\nabla_\mu W^\mu + \frac{n-2}{2} W^2 = g^{\mu\nu} \mathcal{F}_{\mu\nu}[W] \to \frac{1}{2(n-1)} R = g^{\mu\nu}\ A_{\mu\nu}$$

(Ricci gauging)

Weyl invariance in the matter sector then means that

$$S\left(\phi, \mathcal{F}_{\mu\nu}[W] \to A_{\mu\nu}\right) = S((\phi^{\omega}, A_{\mu\nu} + \mathcal{F}_{\mu\nu}[\omega])$$

In flat space the Schouten tensor vanishes

This means that the theory must be invariant under those transformations for which

$$\mathcal{F}_{\mu\nu}[\omega] = 0$$

Which are precisely the Special conformal transformations

$\therefore$ The theory must be conformal in flat space

Ricci gauging for scalar fields amounts to

$$S = \frac{1}{2}\sum_i \int d(vol)\ \left(\left(\partial_\mu - \frac{n-2}{2}W_\mu\right)\phi_i\left(\partial^\mu - \frac{n-2}{2}W^\mu\right)\phi_i\right) =$$

$$\frac{1}{2}\sum_i \int d(vol)\ \left(\partial_\mu\phi_i\partial^\mu\phi_i + (n-2)\left(\nabla_\rho W^\rho + \frac{n-2}{4}W_\rho W^\rho\right)\phi_i^2\right)$$

This leads exactly to TWG (with a minus sign)

$$\frac{1}{2}\sum_i \int d(vol)\ \left(\partial_\mu\phi_i\partial^\mu\phi_i + \frac{n-2}{2}\frac{1}{2(n-1)}R\phi_i^2\right)$$

The minus sign as before does not indicate ghostly behavior anymore than it does in GR.

Conformal Ward identities

Consider the full partition function of the theory

$$Z\left[J^{\mu\nu}, J\right] \equiv \int \mathcal{D}g_{\mu\nu}\ \mathcal{D}\phi_g\ e^{iS[g_{\mu\nu}\phi_g]+i\int d(vol)(J^{\mu\nu}g_{\mu\nu}+J\ \phi_g)}$$

Gravitational equations of motion

$$\int \mathcal{D}g_{\mu\nu}\ \mathcal{D}\phi\ \frac{1}{i}\ \frac{\delta}{\delta g^{\mu\nu}}\ e^{iS[g_{\mu\nu}\phi_g]+i\int d(vol)(J^{\mu\nu}g_{\mu\nu}+J\ \phi_g)} \equiv \left\langle 0_+ \left| \frac{\delta S}{\delta g_{\mu\nu}} + J^{\mu\nu} \right| 0_- \right\rangle = 0$$

They are always traceless on shell

$$g^{\mu\nu}\left\langle 0_+ \left| \frac{\delta S}{\delta g_{\mu\nu}} \right| 0_- \right\rangle = 0$$

Caveat emptor: Gravitational counterterms in arbitrary gauges

Beta gauge

$$\nabla_\sigma h^{\mu\sigma} = \frac{1+\beta}{2}\nabla^\mu h$$

(Kallosh, Tarasov and Tyutin)

$$W = \frac{1}{48\pi^2}\frac{1}{n-4}\int \sqrt{|g|}\, d^4x \, \left(a\ E_4 + \frac{b}{2}W_4 + \frac{c}{6}R^2\right)$$

$$a \equiv \frac{53}{15} - \frac{b}{2}$$

$$b \equiv 2\gamma^4 - 3\gamma^2 - 4\gamma + \frac{21}{10}$$

$$c \equiv 10\gamma^4 - 3\gamma^2 - 14\gamma + \frac{9}{2}$$

$$\gamma \equiv \frac{\beta}{1-\beta}$$

Gauge dependence of the Conformal Anomaly

In the “unique effective action” of Vilkovisky and de Witt

$$a \equiv \frac{53}{15} - \frac{b}{2}$$

$$b \equiv \frac{121}{10}$$

$$c \equiv \frac{31}{2}$$

(No compelling argument in favor of it)

To conclude, much work needs to be done before TWG and related theories are fully understood.

Consistency of the equations of motion (EM)

Classical EM (through the Einstein frame)

$$R_{\mu\nu}[G] = 0 = R_{\mu\nu} + \frac{2n}{n-2}\frac{\nabla_\mu\phi_g\nabla_\nu\phi_g}{\phi_g^2} - 2\frac{\nabla_\mu\nabla_\nu\phi_g}{\phi_g} - \frac{2}{n-2}\left(\frac{(\nabla\phi_g)^2}{\phi_g^2} + \frac{\nabla^2\phi_g}{\phi_g}\right) g_{\mu\nu} \quad ($$

Classical EM (through TWG)

$$\frac{\delta S^{TWG}}{\delta\phi_g} \equiv -\nabla^2\phi_g + \frac{n-2}{4(n-1)}\, R\, \phi_g = 0$$

$$\frac{8(n-1)}{n-2}\frac{\delta S^{TWG}}{\delta g^{\mu\nu}} \equiv R_{\alpha\beta}\,\phi_g^2 + \frac{2n}{n-2}\nabla_\alpha\phi_g\nabla_\beta\phi_g - 2\phi_g\nabla_\alpha\nabla_\beta\phi_g -$$

$$-\frac{1}{2}\left(R\phi_g^2 + \frac{4}{n-2}(\nabla\phi_g)^2 - 4\phi_g\nabla^2\phi_g\right) g_{\alpha\beta} = 0$$

(Equivalent thanks to Noether; the classical Ward identity)

$$2g^{\mu\nu}\frac{\delta S^{TWG}}{\delta g^{\mu\nu}} + \frac{n-2}{2}\ \phi_g\frac{\delta S^{TWG}}{\delta \phi_g} \equiv -\frac{\delta S}{\delta w(x)} = 0$$

This states the Weyl invariance of TWG

$$\frac{\delta S}{\delta w(x)} \equiv 0$$

Four dimensional Euler density

$$E_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = 0.$$

Closed, but not exact

$$\chi(\text{Schwarzschild}) = 2$$

$$W_4 - E_4 = 2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$$

It is always possible under the integral sign

$$\int d(vol)\ W_4 = \int d(vol)\ 2\left(R_{\mu\nu}^2 - \frac{1}{3}R^2\right) + \int d(vol)\ E_4$$

$$\left\langle g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} \right\rangle = 2 \left. \frac{\delta S_{\rm eff}}{\delta \Omega} \right|_{\Omega=1} \neq 0$$

Conformal Anomaly

Remembering the relatioship between Weyl parameter and graviscalar

$$\Omega \sim \frac{1}{M_p}\sqrt{\frac{n-2}{4(n-1)}}\ \phi_g^{\frac{2}{n-2}}$$

If we were allowed to keep the n-dependence on the fields when writing the lagrangian

$$\frac{1}{M_p^{n-4}}\left(\frac{n-2}{4(n-1)}\right)^{\frac{n-4}{2}}\int d(vol)\ \phi_g{}^{\frac{2(n-4)}{n-2}}\ W_4$$

This lagrangian is Weyl invariant for any dimension

Englert et al

In the broken phase it is possible to represent the graviscalar field in an exponential way

$$\phi_g \equiv -2\sqrt{\frac{2(n-1)}{n-2}} M_p^{\frac{n-2}{2}} \; e^{-\sqrt{\frac{n-2}{4(n-1)}}\sigma}$$

This can in a sense be viewed as a change in the measure.

Riemannian measure $\sqrt{|g|}d^4x$

Weyl measure

Standard Lore:

The functional form of the lagrangian must be fixed in, for example, n=4 dimensions BEFORE dimensional regularization is applied

Then

$$\left\langle g^{\mu\nu}\,\frac{\delta S}{\delta g^{\mu\nu}}\right\rangle = 2\,\frac{\delta S_{\rm eff}}{\delta\Omega}\bigg|_{\Omega=1} = \frac{1}{\pi^2}\int d^4x\sqrt{|g|}\,\frac{7}{320}W_4$$