Energy-momentum inequalities for asymptotically anti-de Sitter spacetimes

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Abstract

This talk is based on the joint work with WANG Yaohua and ZHANG Xiao in arXiv:1207.2914.

We establish a new positive energy theorem for certain asymptotically anti-de Sitter initial data sets.

Our energy-momentum inequalities refine the one in the general coordinates pulled back from that obtained in the special "center of AdS mass" coordinates by Chruściel-Maerten-Tod.

Spacetimes

A spacetime is a 4-dimensional Lorentzian manifold $(\mathbf{L}^{3,1}, \mathbf{g})$ which satisfies the Einstein field equations

$$\mathbf{R}_{\alpha\beta} - \frac{\mathbf{R}}{2}\, \mathbf{g}_{\alpha\beta} = \mathbf{T}_{\alpha\beta},$$

where $\mathbf{R}_{\alpha\beta}$ is the Ricci curvature of $\mathbf{g}$ and $\mathbf{R}$ is the scalar curvature of $\mathbf{g}$, $\mathbf{T}_{\alpha\beta}$ is the energy-momentum tensor of matter.

Initial data sets

Initial data set (M, g, h)*:* M is a spacelike hypersurface in $\mathbf{L}^{3,1}$, g is the Riemannian metric of M and h is the second fundamental form of M.

Dominant energy condition

The meaning of "local mass density is nonnegative".

The spacetime $(\mathbf{L}^{3,1}, \mathbf{g})$ satisfies *the dominant energy condition* if, for any timelike vector W,
$\mathbf{T}_{uv} W^u W^v \geq 0$;
$\mathbf{T}^{uv} W_u$ is a non-spacelike vector.

Choose a frame such that e_0 is timelike and e_i is spacelike, we have

$$T^{00} \geq |T^{\alpha\beta}|, \quad T^{00} \geq \sqrt{T^{0i} T^{0}{}_{i}}.$$

Asymptotic flatness

The meaning of "isolated physical systems".

An initial data set (M, g, h) is *asymptotically flat* if there is a compact set $K \subset M$ such that

$$M \setminus K \cong M_c.$$

$M_c \cong \mathbb{R}^3 \setminus B_r$ is called the "end" of M, where B_r is the closed ball of radius r with center at the coordinate origin.

On this end, as $r \to \infty$, g and h have the following asymptotic behaviors

$$g_{ij} = \delta_{ij} + O(\frac{1}{r}),\ \partial_k g_{ij} = O(\frac{1}{r^2}),\ \partial_l \partial_k g_{ij} = O(\frac{1}{r^3}),$$

$$h_{ij} = O(\frac{1}{r^2}),\ \partial_k h_{ij} = O(\frac{1}{r^3})$$

where $\{x^i\}$ is the Euclidean coordinates of $\mathbb{R}^3$.

ADM total energy-momentum

Arnowitt-Deser-Misner(1961): Let (M, g, h) be an asymptotically flat initial data set. Let S_r be the sphere of radius r in the end, and $1 \leq k \leq 3$.

- The ADM total energy of the end is defined as

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) * dx^i.$$

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- The ADM total linear momentum of the end is defined as

$$P_k = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (h_{ki} - g_{ki} tr_g(h)) * dx^i.$$

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- The ADM total energy-momentum are geometric invariants which are independent on the choice of asymptotic coordinates (Bartnik, Chruściel).

- Physically, E is interpreted as the total energy, and P_k is interpreted as the total "translation" of an isolated gravitational system contributed from both matter and gravity.

Schwarzschild spacetime

- $$ds^2 = -\frac{(1-\frac{m}{2\rho})^2}{(1+\frac{m}{2\rho})^2}dt^2 + \left(1+\frac{m}{2\rho}\right)^4\left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

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- Take time slice $M^3 = \{t = 0\}$, then the induced 3-metric, which is asymptotically flat, reads
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- The linear momentum $P_k = 0$.
- Its ADM mass $m_{ADM} = m$.

Positive mass theorem

Theorem (1979, Schoen-Yau; 1981, Witten): *If the spacetime* $(\mathbf{L}^{3,1}, \mathbf{g})$ *satisfies the dominant energy condition, then, for asymptotically flat initial data set* (M, g, h)*,*

$$E \geq \sqrt{P_k P^k}.$$

Equality implies that $\mathbf{L}^{3,1}$ *is flat along* M*.*

Anti de-Sitter spacetime

- The anti-de Sitter spacetime is indeed the hyperboloid

$$\eta_{\alpha\beta} y^\alpha y^\beta = \frac{3}{\Lambda}, \qquad \Lambda = -3\kappa^2 \ (\kappa > 0) \tag{1}$$

in $\mathbb{R}^{3,2}$ equipped with the metric

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- Under the coordinates

$$y^0 = \frac{1}{\kappa}\cos(\kappa t)\cosh(\kappa r), \ y^i = \frac{1}{\kappa}\sinh(\kappa r) n^i,$$

$$y^4 = \frac{1}{\kappa}\sin(\kappa t)\cosh(\kappa r),$$

where $n^1 = \sin\theta\cos\psi$, $n^2 = \sin\theta\sin\psi$, $n^3 = \cos\theta$, the induced metric is

$$\widetilde{g}_{AdS} = -\cosh^2(\kappa r) dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2}\left(d\theta^2 + \sin^2\theta d\psi^2\right). \tag{2}$$

Anti de-Sitter spacetime

- The are ten Killing vectors

$$U_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} - y_\beta \frac{\partial}{\partial y^\alpha} \tag{3}$$

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- The AdS spacetime is a vacuum solution to the Einstein field equation with a NEGATIVE cosmological constant.

$$\mathbf{R}_{\alpha\beta} - \frac{\mathbf{R}}{2}\,\mathbf{g}_{\alpha\beta} + \Lambda \mathbf{g}_{\alpha\beta} = 0, \quad \Lambda = -3\kappa^2 \ (\kappa > 0).$$

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- When the cosmological constant is negative and spacetimes are asymptotically anti-de Sitter, initial data sets are asymptotically hyperbolic and the second fundamental forms are asymptotic to zero, and some special cases of the positive energy theorem were proved in mathematically rigorous ways by X. Wang, P. Chruściel-M. Herzlich, X. Zhang, D. Maerten, N. Xie-X. Zhang, etc.

Asymptotically anti-de Sitter initial data

- Let the coframe of (2) be

$$\breve{e}^0 = \cosh(\kappa r)dt,\ \breve{e}^1 = dr,\ \breve{e}^2 = \frac{\sinh(\kappa r)}{\kappa}d\theta,\ \breve{e}^3 = \frac{\sinh(\kappa r)\sin\theta}{\kappa}d\psi$$

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- Let $(N, \widetilde{g})$ be a spacetime with negative cosmological constant Λ, and $\widetilde{g}$ satisfies the Einstein field equations

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- Suppose that the stress-energy tensor T satisfies the dominant energy condition

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad T_{00} \geq |T_{\alpha\beta}|. \tag{5}$$

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- (M, g, h) is said to be asymptotically anti-de Sitter of order $\tau > \frac{3}{2}$ if
- (1) There is a compact set $K \subset M$ such that

$$M \setminus K \cong M_c.$$

$M_c \cong \mathbb{R}^3 \setminus B_r$ is called the "end" of M, where B_r is the closed ball of radius r with center at the coordinate origin.

Asymptotically anti-de Sitter initial data

- (2) Under this diffeomorphism, the metric $g_{ij} = g(\breve{e}_i, \breve{e}_j)$ on this end is of the form

$$g_{ij} = \delta_{ij} + a_{ij}$$

where a_{ij} satisfies

$$a_{ij} = O(e^{-\tau\kappa r}),\ \breve{\nabla}_k a_{ij} = O(e^{-\tau\kappa r}),\ \breve{\nabla}_l \breve{\nabla}_k a_{ij} = O(e^{-\tau\kappa r}); \tag{6}$$

and the second fundamental form $h_{ij} = h(\breve{e}_i, \breve{e}_j)$ satisfies

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- (3) There exists a distance function ρ_z such that $T_{00} e^{\kappa\rho_z}$, $T_{0i} e^{\kappa\rho_z} \in L^1(M)$. Here $\breve{\nabla}$ denotes the Levi-Civita connection with respect to the hyperbolic metric
$$\breve{g} = dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2}\left(d\theta^2 + \sin^2\theta d\psi^2\right).$$

Energy-momenta

- Set

$$\mathcal{E}_i = \breve{\nabla}^j g_{ij} - \breve{\nabla}_i tr_{\breve{g}}(g) - \kappa(a_{1i} - g_{1i} tr_{\breve{g}}(a)), \quad \mathcal{P}_{ki} = h_{ki} - g_{ki} tr_{\breve{g}}(h).$$

Denote also by $U_{\alpha\beta}$ the restrictions of the Killing vectors (3) on the $t-$slice. We denote

$$\breve{G}^{ijkl} = \frac{1}{2}\sqrt{\breve{g}}\left(\breve{g}^{ik}\breve{g}^{jl} + \breve{g}^{il}\breve{g}^{jk} - 2\breve{g}^{ij}\breve{g}^{kl}\right).$$

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- Henneaux and Teitelboim defined the energy-momenta as

$$J_{ab} = \lim_{r\to\infty} \int_{S_r} \breve{G}^{ijkl}[U^{\perp}_{ab}\breve{\nabla}_j g_{kl} - \breve{\nabla}_j U^{\perp}_{ab} a_{kl}]dS_i + \lim_{r\to\infty} \int_{S_r} 2U^{(k)}_{ab}\pi_k{}^i dS_i,$$

where $\pi_k{}^i = \mathcal{P}_k{}^i$. In the orthonormal frame of (2),

$$J_{ab} = \lim_{r\to\infty} \int_{S_r} \breve{G}^{1jkl}[U^{(0)}_{ab}\breve{\nabla}_j g_{kl} - \breve{\nabla}_j U^{(0)}_{ab} a_{kl}]\breve{\omega} + \lim_{r\to\infty} \int_{S_r} 2U^{(k)}_{ab}\mathcal{P}_{k1}\breve{\omega}.$$

For the convenience of the statement of our main theorem, we introduce the following notions.

$$
\begin{aligned}
E_0 &= \frac{\kappa}{16\pi} \lim_{r\to\infty} \int_{S_r} \mathcal{E}_1 U_{40}^{(0)} \breve{\omega}, \\
c_i &= \frac{\kappa}{16\pi} \lim_{r\to\infty} \int_{S_r} \mathcal{E}_1 U_{i4}^{(0)} \breve{\omega} + \frac{\kappa}{8\pi} \sum_{j=2}^{3} \lim_{r\to\infty} \int_{S_r} \mathcal{P}_{j1} U_{i4}^{(j)} \breve{\omega}, \\
c_i' &= \frac{\kappa}{16\pi} \lim_{r\to\infty} \int_{S_r} \mathcal{E}_1 U_{i0}^{(0)} \breve{\omega} + \frac{\kappa}{8\pi} \sum_{j=2}^{3} \lim_{r\to\infty} \int_{S_r} \mathcal{P}_{j1} U_{i0}^{(j)} \breve{\omega}, \\
J_i &= \frac{\kappa}{8\pi} \sum_{j=2}^{3} \lim_{r\to\infty} \int_{S_r} \mathcal{P}_{j1} V_i^{(j)} \breve{\omega},
\end{aligned}
\tag{8}
$$

where $\breve{\omega} = \breve{e}^2 \wedge \breve{e}^3$, $U_{\alpha\beta} = U_{\alpha\beta}^{(\gamma)} \breve{e}_\gamma$, $V_i = \varepsilon_{ijk} U_{jk}$.

Energy-momenta

The above quantities we denote here are closely related to Henneaux and Teitelboim's total energy-momenta. In fact one has

$$E_0 = \frac{\kappa}{16\pi} J_{40}, \quad c_i = \frac{\kappa}{16\pi} J_{i4}, \quad c_i' = \frac{\kappa}{16\pi} J_{i0}, \quad J_i = \frac{\kappa}{16\pi} \varepsilon_{ijk} J_{jk}.$$

A new positive energy theorem

Theorem (Wang-Xie-Zhang) Let (M, g, h) be a 3-dimensional asymptotically anti-de Sitter initial data set in spacetime $(N, \widetilde{g})$. Suppose $(N, \widetilde{g})$ satisfies the dominant energy condition. Then

$$\begin{aligned} &(i) \;\; E_0 \geq \Big(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}' \times \mathbf{J}| + 2C_1^{\frac{1}{2}}\Big)^{\frac{1}{2}} \geq |\mathbf{c}|; \\ &(ii) \; E_0 \geq \Big(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 - 2|\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}} + 2C_2^{\frac{1}{2}}\Big)^{\frac{1}{2}} \geq \Big(\frac{|\mathbf{c}'|^2 + |\mathbf{J}|^2}{2}\Big)^{\frac{1}{2}}, \end{aligned} \tag{9}$$

A new positive energy theorem

where

$$C_1 = \max\left\{\left[B + |\mathbf{c}' \times \mathbf{J}|^2 - |\mathbf{c}' \times \mathbf{J}|\left(2|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2\right)\right], 0\right\},$$
$$C_2 = \max\left\{\left[B + |\mathbf{c}|^2|\mathbf{c}' \times \mathbf{J}| - |\mathbf{c}||\mathbf{c}' \times \mathbf{J}|^{\frac{1}{2}}\left(|\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\mathbf{J}|^2 + |\mathbf{c}' \times \mathbf{J}|\right)\right], 0\right\}.$$

Here $\mathbf{c} = (c_1, c_2, c_3)$, $\mathbf{c}' = (c_1', c_2', c_3')$, $\mathbf{J} = (J_1, J_2, J_3) = \vec{j}$ and

$$B = |\mathbf{c} \times \mathbf{c}'|^2 + |\mathbf{c} \times \mathbf{J}|^2 + |\mathbf{c}' \times \mathbf{J}|^2. \qquad (10)$$

A new positive energy theorem

If $E_0 = 0$, then one has $Q = 0$, and $(N, \widetilde{g})$ is anti-de Sitter along M. Here the energy-momentum matrix Q is

$$\begin{aligned} Q &= \begin{pmatrix} E & L \\ \bar{L}^t & \hat{E} \end{pmatrix}, \qquad E = \begin{pmatrix} E_0 - c_3 & c_1 - \sqrt{-1}c_2 \\ c_1 + \sqrt{-1}c_2 & E_0 + c_3 \end{pmatrix}, \\ L &= \begin{pmatrix} l_1 & l_2 \\ l_3 & -l_1 \end{pmatrix}, \qquad \hat{E} = \begin{pmatrix} E_0 + c_3 & -c_1 + \sqrt{-1}c_2 \\ -c_1 - \sqrt{-1}c_2 & E_0 - c_3 \end{pmatrix}, \end{aligned} \tag{11}$$

$l_1 = c_3' - \sqrt{-1}J_3$, $l_2 = -c_1' + J_2 + \sqrt{-1}(c_2' + J_1)$,
$l_3 = -c_1' - J_2 - \sqrt{-1}(c_2' - J_1)$.

Chruściel-Maerten-Tod's work

- Chruściel, Maerten, and Tod provided definitions of the total energy $m_{(\nu)}$ ($\nu = 0, 1, 2, 3$), the rest-frame angular momentum $j_{(i)}$ and the center of mass $c_{(i)}$, $i = 1, 2, 3$.

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- They also pointed out that one can make $SO(3,1)$ coordinate transformations for the 0-slice in the above AdS spacetime such that

$$m_{(1)} = m_{(2)} = m_{(3)} = c_{(2)} = j_{(1)} = j_{(2)} = 0, \qquad (12)$$

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- and they proved the energy-momentum inequality

$$m_{(0)} \geq \sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|} \qquad (13)$$

in this new coordinate system. Here $\vec{m} = (m_{(1)}, m_{(2)}, m_{(3)})$, $\vec{c} = (c_{(1)}, c_{(2)}, c_{(3)})$ and $\vec{j} = (j_{(1)}, j_{(2)}, j_{(3)})$.

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- We refer to the coordinates satisfying (12) as the "center of AdS mass" coordinates.

The relation between Chruściel-Maerten-Tod's quantities and ours

We obtain the following relations between our total energy-momenta and $m_{(\nu)}$, $c_{(i)}$, $J_{(i)(j)}$ and $j_{(k)}$

$$E_0 = m_{(0)}, \ \ c_i = -m_{(i)} \cos \kappa t + c_{(i)} \sin \kappa t, \ \ c_i' = m_{(i)} \sin \kappa t + c_{(i)} \cos \kappa t,$$
$$J_k = \varepsilon_{kij} J_{(i)(j)} = j_{(k)}, \quad \sum_{1 \leq i \leq 3} c_i^2 + {c'}_i^2 = \sum_{1 \leq i \leq 3} m_{(i)}^2 + c_{(i)}^2. \tag{14}$$

Remarks

If $E_0 > 0$, then, implicitly, $\mathbf{c}'$ and $\mathbf{J}$ can be chosen freely in (9)(i) and $\mathbf{c}$ can be chosen freely in (9)(ii). In particular, we can choose $\mathbf{c}' = 0$ in (9)(i) or $\mathbf{c} = 0$ in (9)(ii), then, without assuming (12), (9)(i) at $t = \frac{\pi}{2\kappa}$ or (9)(ii) at $t = 0$ reduces to (13). Moreover, (9)(i) at $t = 0$ or (9)(ii) at $t = \frac{\pi}{2\kappa}$ gives

$$m_{(0)} \geq \sqrt{|\vec{m}|^2 + |\vec{j}|^2 + 2|\vec{m} \times \vec{j}|}. \qquad (15)$$

This indicates that $\vec{m}$ and $\vec{c}$ play the same role in physics.

Remarks

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- And we also show that the determinant of the energy-momenta matrix $\det Q$ is invariant under such a $SO(3,1)$ coordinate transformation.
- We find that Chruściel-Maerten-Tod's energy-momentum inequality (13) is the same as the nonnegativity of the determinant of the energy-momenta matrix Q transforming back to the general non-center of AdS mass coordinates.

Thanks!

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