

# Point Massive Particle in General Relativity

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## Notations

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$\mathbb{M} \approx \mathbb{R}^4$  - topologically trivial manifold (space-time)

$x^\alpha$ ,  $\alpha = 0, 1, 2, 3$  - Cartesian coordinates

$g_{\alpha\beta}(x)$ ,  $\text{sign } g_{\alpha\beta} = (+---)$  - Lorentzian signature metric

$q^\alpha(\tau)$  - worldline of a point particle  $\tau$  - parameter along worldline

## The action

$$S = \frac{1}{16\pi} \int dx \sqrt{|g|} R - M \int d\tau \sqrt{\dot{q}^\alpha \dot{q}^\beta g_{\alpha\beta}}$$

$R(g)$  - scalar curvature

$M$  - mass of a point particle

$\dot{q}^\alpha := \frac{dq^\alpha}{d\tau}$  - velocity of a particle

## Equations of motion

Einstein's equations: 
$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = -\frac{1}{2} T^{\alpha\beta} \quad (*)$$

Geodesic equations: 
$$\left( \ddot{q}^\alpha + \Gamma_{\beta\gamma}^\alpha \Big|_{x=q} \dot{q}^\beta \dot{q}^\gamma \right) g_{\alpha\delta} = 0$$

where  $T^{\alpha\beta} = \frac{16\pi M \dot{q}^\alpha \dot{q}^\beta}{\sqrt{|g|} \dot{q}^0} \delta(\mathbf{x} - \mathbf{q})$  - energy-momentum tensor

$\delta(\mathbf{x} - \mathbf{q}) := \delta(x^1 - q^1) \delta(x^2 - q^2) \delta(x^3 - q^3)$  - three-dimensional  $\delta$ -function

$R_{\alpha\beta}$  - Ricci tensor

$\Gamma_{\beta\gamma}^\alpha$  - Christoffel's symbols

The Schwarzschild solution does not satisfy equations of motion (\*)

$$ds^2 = \left( 1 - \frac{2M}{\rho} \right) dt^2 - \frac{d\rho^2}{\left( 1 - \frac{2M}{\rho} \right)} - \rho^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

**Where is the  $\delta$ -function ?**

## Point mass in General Relativity

$$S = \frac{1}{16\pi} \int dx \sqrt{|g|} R - M \int d\tau \sqrt{\dot{q}^\alpha \dot{q}^\beta g_{\alpha\beta}}$$

Canonical formulation

$$g_{\alpha\beta} = \begin{pmatrix} N^2 + N^\rho N_\rho & N_\nu \\ N_\mu & g_{\mu\nu} \end{pmatrix} \quad \text{- ADM parameterization of the metric}$$

$\alpha = (0, \mu), \quad \mu = 1, 2, 2$

$N$  - lapse function

$N_\mu$  - shift functions

$$N^\rho := \hat{g}^{\rho\sigma} N_\sigma \quad \hat{g}^{\mu\nu} g_{\nu\rho} = \delta_\rho^\mu$$

$(g_{\mu\nu}, p^{\mu\nu}), (q^\mu, p_\mu)$  - canonically conjugate pairs of dynamical variables

$N, N^\mu$  - Lagrange multipliers

Time gauge:  $\tau = q^0 = x^0 := t$

Notations:  $p := p^{\mu\nu} g_{\mu\nu}, \hat{p}^2 := \hat{g}^{\mu\nu} p_\mu p_\nu$

$\hat{\nabla}_\mu, \hat{\Delta} := \hat{\nabla}^\mu \hat{\nabla}_\mu$  - three-dimensional covariant derivative and Laplace-Beltrami operator

# Hamiltonian equations of motion

Constraints:

$$H_{\perp} = \frac{1}{\hat{e}} (p^{\mu\nu} p_{\mu\nu} - p^2) - \hat{e} \hat{R} + \sqrt{M^2 + \hat{p}^2} \delta(\mathbf{x} - \mathbf{q}) = 0$$

$$\hat{e} := \sqrt{|\det g_{\mu\nu}|}$$

$$H_{\mu} = -2\hat{\nabla}^{\nu} p_{\nu\mu} - p_{\mu} \delta(\mathbf{x} - \mathbf{q}) = 0$$

Dynamical equations:

$$\dot{g}_{\mu\nu} = \frac{2N}{\hat{e}} p_{\mu\nu} - \frac{N}{\hat{e}} g_{\mu\nu} + \hat{\nabla}_{\mu} N_{\nu} + \hat{\nabla}_{\nu} N_{\mu}$$

$$\dot{p}^{\mu\nu} = \frac{N}{2\hat{e}} \hat{g}^{\mu\nu} \left( p^{\rho\sigma} p_{\rho\sigma} - \frac{1}{2} p^2 \right) - \frac{2N}{\hat{e}} \left( p^{\mu\rho} p^{\nu}_{\rho} - \frac{1}{2} p^{\mu\nu} p \right) +$$

$$+ \hat{e} (\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}^{\mu} \hat{\nabla}^{\nu} N) - \hat{e} N \left( \hat{R}^{\mu\nu} - \frac{1}{2} \hat{g}^{\mu\nu} \hat{R} \right) - p^{\mu\rho} \hat{\nabla}_{\rho} N^{\nu} - p^{\mu\rho} \hat{\nabla}_{\rho} N^{\nu} +$$

$$+ \hat{\nabla}_{\rho} (N^{\rho} p^{\mu\nu}) - \frac{N p^{\mu} p^{\nu}}{2\sqrt{M^2 + \hat{p}^2}} \delta(\mathbf{x} - \mathbf{q})$$

Geodesics:

$$\dot{q}^{\mu} = - \frac{N}{\sqrt{M^2 + \hat{p}^2}} \Big|_{\mathbf{x}=\mathbf{q}} p^{\mu} - N^{\mu} \Big|_{\mathbf{x}=\mathbf{q}}$$

$$\dot{p}_{\mu} = -\partial_{\mu} \left[ N \sqrt{M^2 + \hat{p}^2} - N^{\nu} p_{\nu} \right]_{\mathbf{x}=\mathbf{q}}$$

## Solution of the equations of motion

Spherical coordinate system  $t, r, \vartheta, \varphi$  with particle located at the origin  $\mathbf{q} = 0$

Staticity:  $g_{\alpha\beta} = g_{\alpha\beta}(\mathbf{x}), \quad p^{\mu\nu} = 0, \quad p_{\mu} = 0$

Spherical symmetry:  $g_{\mu\nu} = g_{\mu\nu}(r) \quad N = N(r) \quad N_{,\mu} = 0$

Equations of motion:

$$\begin{aligned} -\hat{e}\hat{R} + 16\pi M \delta(\mathbf{x}) &= 0 \\ \hat{e}(\hat{\Delta}N\hat{g}^{\mu\nu} - \hat{\nabla}_{\mu}\hat{\nabla}_{\nu}N) - \hat{e}N\left(\hat{R}^{\mu\nu} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{R}\right) &= 0 \\ \partial_{\mu}N|_{\mathbf{x}=0} &= 0 \end{aligned}$$

## The main equation

$$\hat{e}\hat{R} = 16\pi M \delta(\mathbf{x})$$

$$\hat{e} := \sqrt{|\det g_{\mu\nu}|}$$

$\hat{R}$  - three-dimensional scalar curvature

$\delta(\mathbf{x}) := \delta(x^1)\delta(x^2)\delta(x^3)$  - three-dimensional  $\delta$ -function

We are seeking spherically symmetric solution  $g_{\mu\nu} = -f^2\delta_{\mu\nu}$  - ansatz

$\delta_{\mu\nu} = \text{diag}(+++)$  - Euclidean metric

$$\Delta f - \frac{\partial f^2}{2f} = -4\pi M \delta(\mathbf{x})$$

$\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$  - flat space Laplacian

$\partial f^2 := \delta^{\mu\nu} \partial_\mu f \partial_\nu f$  - notation

$\mathcal{D}(\mathbb{R}^3)$  - test functions (smooth functions with compact support)

$\mathcal{D}'(\mathbb{R}^3)$  - space of generalized functions (distributions)

Theorem. Function

$$f = 1 + \frac{M}{r} + \frac{M^2}{4r^2} = \left(1 + \frac{M}{2r}\right)^2$$

satisfies equation

$$\Delta f - \frac{\partial f^2}{2f} = -4\pi M \delta(\mathbf{x}) \quad (*)$$

which solution is understood in a generalized sense after integration with a test function.

Proof.

$f \in \mathcal{D}'(\mathbb{R}^3)$  - locally integrable function

$(r^\lambda, \varphi) := \int_{\mathbb{R}^3} dx r^\lambda \varphi$  - analytic functional  $\lambda \in \mathbb{C}$  except poles at  $\lambda = -3, -5, -7, \dots$   
Gel'fand, Shilov. Vol.1 (1958)

$$\frac{\partial f^2}{2f} = \frac{M^2}{2r^4} \quad (*) \implies \Delta \left(1 + \frac{M}{r} + \frac{M^2}{4r^2}\right) - \frac{M^2}{2r^4} = -4\pi M \delta(\mathbf{x})$$

$$\Delta \left(1 + \frac{M}{r}\right) = \Delta \frac{M}{r} = -4\pi M \delta(\mathbf{x}) \quad \text{- fundamental solution}$$

$$\Delta \frac{M^2}{4r^2} - \frac{M^2}{2r^4} = 0$$

$$\Delta r^{\eta+2} = (\eta+2)(\eta+3)r^\eta, \quad \eta \in \mathbb{C}, \operatorname{re} \eta > 0 \quad \text{analytic continuation to } \eta = -4 \quad 7$$

Theorem. Let three-dimensional metric be  $g_{\mu\nu} = -\left(1 + \frac{M}{2r}\right)^2 \delta_{\mu\nu}$

Then the lapse function  $N = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}$  satisfy equation

$$\hat{e}(\hat{\Delta}N\hat{g}^{\mu\nu} - \hat{\nabla}^{\mu}\hat{\nabla}^{\nu}N) - \hat{e}N\left(\hat{R}^{\mu\nu} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{R}\right) = 0$$

in a generalized sense.

$$N(r) > 0, \quad \frac{M}{2} < r < \infty$$

$N(r)$  - locally integrable function

$$N(r) = 0, \quad r = \frac{M}{2}$$

$$N \in \mathcal{D}'(\mathbb{R}^3)$$

$$N(r) < 0, \quad 0 < r < \frac{M}{2}$$

$$N(0) = -1.$$



## Geodesic equations

$$\partial_r N = \frac{M}{r^2} \frac{1}{\left(1 + \frac{M}{2r}\right)^2} \neq 0 \quad - \text{geodesic equation is not fulfilled}$$

$$\left\langle \frac{M}{r^3} \frac{x_\mu}{\left(1 + \frac{M}{2r}\right)^2} \right\rangle = 0 \quad - \text{after averaging over sphere}$$

## Newton's mechanics

$$m\vec{a} = \vec{F} \quad F_\mu = \partial_\mu \frac{1}{r} = \frac{x_\mu}{r^3} \quad - \text{diverges at } r \rightarrow 0$$

Equations of motions are not fulfilled

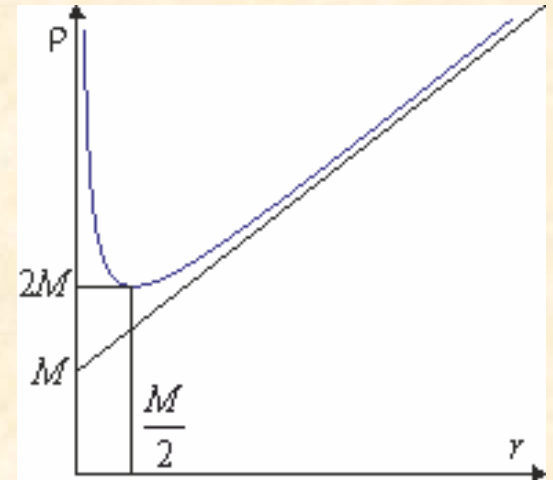
$$-\frac{x_\mu}{r^3} = 0 \quad - \text{after averaging over sphere}$$

## Relation to the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2M}{\rho}\right) dt^2 - \frac{d\rho^2}{1 - \frac{2M}{\rho}} - \rho^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- the Schwarzschild metric in Schwarzschild coordinates, white and black holes, no point particle

$$\rho = r \left(1 + \frac{M}{2r}\right)^2 \quad \text{- change of radial coordinate}$$



$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 \left[ dr^2 + (d\vartheta^2 + \sin^2\vartheta d\varphi^2) \right]$$

- the Schwarzschild metric in isotropic coordinates, point particle, no black hole

## Schwarzschild metric in isotropic coordinates

$$ds^2 = \left( \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 dt^2 - \left( 1 + \frac{M}{2r} \right)^4 \left[ dr^2 + (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right]$$

$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi$$

$\mathbb{M} \approx \mathbb{R}^4$  - topologically trivial space-time

Asymptotic flatness at  $r \rightarrow \infty$

All components are smooth for  $0 < r < \infty$

$$\det g_{\alpha\beta} = - \left( 1 - \frac{M}{2r} \right)^2 \left( 1 + \frac{M}{2r} \right)^{10} r^4 \sin^2 \vartheta$$

Metric is degenerate on the sphere  $r_* = \frac{M}{2} \Leftrightarrow \rho_s = 2M$

↑  
Schwarzschild radius (horizon)

Asymptotic at large distances:  $g_{00} \simeq 1 - \frac{2M}{r} \quad r \rightarrow \infty$

$\Rightarrow$  Newton's law

## The Einstein-Rosen Bridge

Coordinate transformation  $\rho \rightarrow u$ :  $\frac{1}{2}u^2 = \rho - 2M$

$$ds^2 = \frac{u^2}{u^2 + 4M} dt^2 - (u^2 + 4M) du^2 - \frac{1}{4}(u^2 + 4M)(d\vartheta^2 + \sin^2 d\varphi^2)$$

Two copies of external Schwarzschild solution are mapped onto  $u > 0$  and  $u < 0$

The metric is degenerate at  $u = 0$

Transformation to isotropic coordinates:  $u = \sqrt{2r} \left( 1 - \frac{M}{2r} \right)$

$$u > 0 \iff \frac{M}{2} < r < \infty$$

$$u < 0 \iff 0 < r < \frac{M}{2}$$

The Schwarzschild metric in isotropic coordinates is globally isometric to Einstein-Rosen bridge

## Conclusion

This is the gravitational field around  
massive point particle of mass  $M$

$$ds^2 = \left( \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 dt^2 - \left( 1 + \frac{M}{2r} \right)^4 \left[ dr^2 + (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right]$$

$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi$$

$\mathbb{M} \approx \mathbb{R}^4$  - topologically trivial and asymptotically flat

The space-time is geodesically complete at  $r \rightarrow 0, \infty$

and incomplete at  $r \rightarrow \frac{M}{2}$

$r > \frac{M}{2}$  - attraction

$0 < r < \frac{M}{2}$  - repulsion